Covariant Superspace Approaches to $\mathcal{N} = 2$ Supergravity 39

$\mathcal{N} = 2$ 超引力的协变超空间方法 39

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Abstract

摘要

We provide a unified description of the three covariant superspace approaches to $\mathcal{N} = 2$ conformal supergravity in three dimensions: (i) conformal superspace, (ii) $U(2)$ superspace, and (iii) $SU(2)$ superspace. Each of them can be used to formulate general supergravity-matter systems, although conformal superspace has the largest structure group and is intimately related to the superconformal tensor calculus. We review the structure of covariant projective multiplets and demonstrate how they are used to describe pure and matter-coupled supergravity, including locally superconformal off-shell sigma models. Higher-derivative invariants, topological invariants, and super-Weyl anomalies are also briefly discussed.

我们针对三维空间中的 $\mathcal{N}=2$ 共形超引力，对三种协变超空间方法给出统一描述:(i) 共形超空间；(ii) $U(2)$ 超空间；(iii) $SU(2)$ 超空间。尽管共形超空间拥有最大的结构群，且与超共形张量演算密切相关，但上述三种方法均可用于构造一般的超引力-物质系统。我们综述了协变投影多重态的结构，说明了如何利用它们描述纯超引力与耦合物质的超引力，包括局域超共形脱壳 sigma 模型。文中还简要讨论了高阶导数不变量、拓扑不变量与超外尔反常。

Keywords

关键词

Superconformal symmetry - Supergravity - Superspace - Projective multiplets

超共形对称 - 超引力 - 超空间 - 投影多重态

To Daniel Butter with gratitude and admiration

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Introduction

引言

Pure $\mathcal{N}=2$ supergravity in four dimensions was constructed by Ferrara and van Nieuwenhuizen in 1976 [1], some 6 months after the creation of $\mathcal{N}=1$ supergravity [2, 3] . It fulfilled Einstein's dream of unifying gravity and electromagnetism, albeit using a symmetry principle that was not known to Einstein - local supersymmetry.

四维纯 $\mathcal{N}=2$ 超引力由 Ferrara 与 van Nieuwenhuizen 于 1976 年构造完成 [1]，距 $\mathcal{N}=1$ 超引力 [2, 3] 提出仅约六个月。它实现了爱因斯坦统一引力与电磁学的梦想，只是所用的对称原理——局部超对称——是爱因斯坦生前未曾知晓的。

In 1979, Fradkin and Vasiliev [4] and, independently, de Wit and van Holten [5] proposed an off-shell formulation for linearized $\mathcal{N}=2$ supergravity. Shortly thereafter, these linearized results were extended to the first off-shell formulation for $\mathcal{N}=2$ supergravity [6, 7] . In [7] de Wit, van Holten, and Van Proeyen made use of the so-called $\mathcal{N}=2$ superconformal tensor calculus, a natural extension of the $\mathcal{N}=1$ superconformal method [8-12]. Since then, the $\mathcal{N}=2$ superconformal tensor calculus of [7] has been further developed [13-15] and applied [16, 17] to derive many important results for $\mathcal{N}=2$ supergravity-matter systems. For comprehensive reviews of this method, see [18, 19].

1979 年，Fradkin 和 Vasiliev[4]，以及独立研究的 de Wit 和 van Holten[5]，提出了线性化 $\mathcal{N}=2$ 超引力的离壳表述。此后不久，这些线性化结果被推广，得到了 $\mathcal{N}=2$ 超引力的首个离壳表述 [6, 7]。文献 [7] 中，de Wit、van Holten 和 Van Proeyen 采用了所谓的 $\mathcal{N}=2$ 超共形张量演算，这是对 $\mathcal{N}=1$ 超共形方法的自然推广 [8-12]。自那以后，文献 [7] 提出的 $\mathcal{N}=2$ 超共形张量演算得到了进一步发展 [13-15]，并被应用于推导 $\mathcal{N}=2$ 超引力-物质系统的诸多重要结果 [16,17]。关于该方法的综合综述可参见 [18,19]。

In parallel with the progress achieved in [4-7], there appeared several works [20-26] devoted to $\mathcal{N}=2$ superfield supergravity. In these works, the component results were recast in a superspace setting. More importantly, these publications pursued an ambitious goal of developing superspace formulations to describe general supergravity-matter systems, including the construction of an off-shell charged hypermultiplet that can be coupled to a $\mathrm{U}(1)$ vector multiplet.

在 [4-7] 取得研究进展的同时，也出现了数篇聚焦 $\mathcal{N}=2$ 超场超引力的工作 [20-26]。这些工作将分量结果重构到了超空间框架中。更重要的是，这些研究怀揣一个宏大目标: 发展超空间表述来描述一般的超引力-物质系统，包括构造可与 $\mathrm{U}(1)$ 矢量多重场耦合的离壳带电超多重场。

Within the superconformal tensor calculus, hypermultiplets are either on-shell or involve a gauged central charge. As is well known, such hypermultiplet realizations cannot be used to provide an off-shell formulation for the most general locally supersymmetric sigma model. It is also known that such a sigma model formulation, if it exists, must use off-shell hypermultiplets possessing an infinite number of auxiliary fields [27-29]. The latter feature makes the off-shell hypermultiplets extremely difficult to work with at the component level, and a superfield setting is required.

在超共形张量演算框架下，超多重场要么是在壳的，要么包含规范中心化荷。众所周知，这类超多重场实现无法用于给出最一般的局部超对称 sigma 模型的离壳表述。同样已知的是，这样的 sigma 模型表述若存在，必须使用包含无穷多个辅助场的离壳超多重场 [27-29]。这一特性导致在分量层面处理离壳超多重场极为困难，因此需要超空间框架。

The problem of constructing an off-shell charged hypermultiplet (in short, the charged hypermultiplet problem) remained unsolved until 1984. Nevertheless, the early works on $\mathcal{N}=2$ superfield supergravity [20-26] have yielded several important results. It suffices to mention the linear multiplet action originally discovered by Sohnius in the rigid supersymmetric case [30]. Since the linear multiplet was lifted to $\mathcal{N}=2$ supergravity [20], and then reformulated [31] within the $\mathcal{N}=2$ superconformal tensor calculus [7, 13, 14] , it has become a universal tool to construct the component actions for supergravity-matter systems.

构造离壳带电超多重场的问题 (简称带电超多重场问题) 直到 1984 年都未得到解决。尽管如此，早期关于 $\mathcal{N}=2$ 超场超引力的研究 [20-26] 仍得到了若干重要结果。仅举一例: 最初由 Sohnius 在刚性超对称情形下发现的线性多重场作用量 [30]。线性多重场被推广到 $\mathcal{N}=2$ 超引力后 [20]，又在 $\mathcal{N}=2$ 超共形张量演算框架内被重新表述 [31] [7, 13, 14]，此后它就成为构造超引力-物质系统分量作用量的通用工具。

The construction of the relaxed hypermultiplet in 1983 [32] was perhaps the pinnacle of conventional $\mathcal{N}=2$ superspace techniques, but it did not solve the charged hypermultiplet problem (There exist infinitely many off-shell formulations for the neutral hypermultiplet [33, 34], in addition to the relaxed hypermultiplet.).

In spite of being off-shell, this hypermultiplet is neutral and cannot couple to a U(1) vector multiplet. It became apparent that the conventional $\mathcal{N} = 2$ superspace $\mathbb{M}^{4|8}$ is not suitable, say, for off-shell σ -model constructions. The correct superspace setting was found in 1983-1984 independently by three groups who pursued somewhat different goals [35-37], which is:

> 1983 年松弛超多重场的构造 [32] 或许是传统 $\mathcal{N} = 2$ 超空间技术的巅峰成果，但它仍未解决带电超多重场问题 (除松弛超多重场外，中性超多重场还存在无穷多种离壳表述 [33, 34])。尽管该超多重场是离壳的，但它是中性的，无法与 U(1) 矢量多重场耦合。人们逐渐意识到，传统 $\mathcal{N} = 2$ 超空间 $\mathbb{M}^{4|8}$ 并不适用于离壳 σ 模型构造。正确的超空间框架在 1983-1984 年由三个追求不同目标的研究组独立发现 [35-37]，即:

$$\mathbb{M}^{4|8} \times \mathbb{C}P^1 = \mathbb{M}^{4|8} \times S^2. \tag{1}$$

This superspace was introduced for the first time by Rosly [35] who used it to derive an interpretation of the $\mathcal{N} = 2$ super Yang-Mills constraints [38] as integrability conditions. Rosly and Schwarz [39] called (1) isotwistor superspace.

> 这种超空间由 Rosly 首次引入 [35]，他借助该超空间将 $\mathcal{N} = 2$ 超杨-米尔斯约束 [38] 阐释为可积性条件。Rosly 与 Schwarz[39] 将 (1) 命名为等扭超空间。

The starting point of the analysis in [35] was the observation that, given an isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$, the set of eight spinor covariant derivatives $D_{\alpha i}$ and $\bar{D}_{\dot\alpha i}$ for $\mathbb{M}^{4|8}$ contains a subset of four operators, $D^{(1)}_{\alpha} := -v^i D_{\alpha i}$ and $\bar{D}^{(1)}_{\dot\alpha} := -v^i \bar{D}_{\dot\alpha i}$, which strictly anticommute with each other. Therefore, one can introduce a new family of supersymmetric multiplets constrained by

> 文献 [35] 分析的出发点是: 给定一个等扭 $v^i \in \mathbb{C}^2 \setminus \{0\}$，对应 $\mathbb{M}^{4|8}$ 的八个旋量协变导数 $D_{\alpha i}$ 与 $\bar{D}_{\dot\alpha i}$ 中存在一个由四个算符 $D^{(1)}_{\alpha} := -v^i D_{\alpha i}$ 和 $\bar{D}^{(1)}_{\dot\alpha} := -v^i \bar{D}_{\dot\alpha i}$ 构成的子集，这些算符两两严格反对易。因此，可以引入一族由下式约束的新超对称多重态:

$$D^{(1)}_{\alpha}\phi = 0,\ \bar{D}^{(1)}_{\dot\alpha}\phi = 0,\ \phi = \phi(z, v, \bar{v}),\ \bar{v}_i := \overline{v^i}. \tag{2}$$

In order for these constraints to be invariant under arbitrary re-scalings of v, ϕ should be homogeneous,

> 为了让这些约束在 v, ϕ 的任意重标度下保持不变，约束应当是齐次的,

$$\phi(z, cv, \bar{c}\bar{v}) = c^{n_+}\bar{c}^{n_-}\phi(z, v, \bar{v}),\ c \in \mathbb{C} \setminus \{0\} \equiv \mathbb{C}^*, \tag{3}$$

for some parameters $n_\pm$ such that $n = n_+ - n_-$ is an integer. Redefining $\phi(z, v, \bar{v}) \to \phi(z, v, \bar{v})/(v^\dagger v)^{n_-}$ allows one to choose $n_- = 0$. Any superfield with the homogeneity property

> 对满足 $n = n_+ - n_-$ 为整数的某些参数 $n_\pm$，重新定义 $\phi(z, v, \bar{v}) \to \phi(z, v, \bar{v})/(v^\dagger v)^{n_-}$ 即可选取 $n_- = 0$。任何满足该齐次性的超场

$$\phi^{(n)}(z, cv, \bar{c}\bar{v}) = c^n\phi^{(n)}(z, v, \bar{v}),\ c \in \mathbb{C}^* \tag{4}$$

is said to have the weight $n \in \mathbb{Z}$. This superfield lives in the superspace (1), since the isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$ is defined modulo the equivalence relation $v^i \sim cv^i$, with $c \in \mathbb{C}^*$, hence it parametrizes $\mathbb{C}P^1$. A weight- n superfield $\phi^{(n)}(z, v, \bar{v})$ is called isotwistor if it obeys the constraints (2).

被称为具有权 $n \in \mathbb{Z}$。该超场存在于超空间 (1) 中，因为等扭 $v^i \in \mathbb{C}^2 \setminus \{0\}$ 是对等价关系 $v^i \sim cv^i$ 取模定义的，其中 $c \in \mathbb{C}^*$ ，因此它参数化了 $\mathbb{C}P^1$ 。满足约束 (2) 的权 n 超场 $\phi^{(n)}(z, v, \bar{v})$ 被称为等扭超场。

A new approach to $\mathcal{N} = 2$ supersymmetric field theory was put forward by Galperin et al. [36]. Using harmonic superspace $\mathbb{M}^{4|8} \times S^2$, they proposed the first off-shell formulation of charged hypermultiplet (the so-called q^+ hypermultiplet) and its self-couplings. Moreover, unconstrained prepotential descriptions of $\mathcal{N} = 2$ super Yang-Mills and supergravity theories were also provided. Since then the harmonic superspace approach has developed into a powerful branch of supersym-metric field theory; see [40] for a review. In the harmonic superspace approach, one deals with those isotwistor superfields $\phi^{(n)}(z, v, \bar{v})$ which are globally defined smooth functions on $\mathbb{C}P^1$. In the literature, they are known as harmonic analytic superfields.

Galperin 等人提出了研究 $\mathcal{N} = 2$ 超对称场论的新方法 [36]。借助调和超空间 $\mathbb{M}^{4|8} \times S^2$ ，他们给出了带电超多重态 (即所谓的 q^+ 超多重态) 的第一个脱壳表述及其自耦合。此外，他们还给出了 $\mathcal{N} = 2$ 超杨-米尔斯理论和超引力的无约束预势描述。自那时起，调和超空间方法发展成为超对称场论的一个强大分支，综述可见 [40]。在调和超空间方法中，我们研究的是在 $\mathbb{C}P^1$ 上整体定义的光滑函数形式的等扭超场 $\phi^{(n)}(z, v, \bar{v})$ 。这类超场在文献中被称为调和解析超场。

Projective superspace $\mathbb{M}^{4|8} \times \mathbb{C}P^1$ was originally employed in [37] to provide a manifestly $\mathcal{N} = 2$ supersymmetric description for the general self-couplings of $\mathcal{N} = 2$ tensor multiplets constructed earlier [41] in terms of $\mathcal{N} = 1$ superfields. Since then, this approach has been extended to include some other interesting multiplets [42, 43]. In particular, a new off-shell formulation for the charged hypermultiplet was derived [42] and used to construct off-shell nonlinear σ -models; see [44,45] for reviews. The name "projective superspace" was coined in 1990 [43]. In the projective-superspace approach, one deals with those isotwistor superfields $\phi^{(n)}(z, v)$ which are holomorphic functions on an open domain of $\mathbb{C}P^1$. In the literature, they are known as projective superfields.

投影超空间 $\mathbb{M}^{4|8} \times \mathbb{C}P^1$ 最初在文献 [37] 中被使用，用于为早前以 $\mathcal{N} = 1$ 超场构造得到的 $\mathcal{N} = 2$ 张量多重态的一般自耦合提供一个明显 $\mathcal{N} = 2$ 超对称描述。此后，该方法被推广，以包含其他有趣的多重态 [42, 43]。特别是，人们推导得到了带电超多重态的一种新离壳表述 [42]，并将其用于构造离壳非线性 σ 模型；综述见 [44,45]。“投影超空间”这一名称是 1990 年创造的 [43]。在投影超空间方法中，我们处理的是满足性质为 $\mathbb{C}P^1$ 开域上全纯函数的同扭矢超场 $\phi^{(n)}(z, v)$ 。在文献中，这类超场被称为投影超场。

Both harmonic and projective superspace make use of the same superspace (1). Without going into technical details, which are spelled out in [46] (see also [45, 47, 48]), they differ in (i) the structure of off-shell supermultiplets used and (ii) the supersymmetric action principle chosen. Due to these conceptual differences, the two approaches prove to be complementary to each other in many respects. In particular, harmonic superspace offers powerful prepotential formulations for $\mathcal{N} = 2$ supergravity [49, 50] (reviewed in [40]) which are similar in spirit to the Ogievetsky-Sokatchev approach to $\mathcal{N} = 1$ supergravity [51]. Projective superspace proves to be ideal for developing covariant geometric formulations for supergravity-matter systems with eight supercharges. The harmonic superspace approach to $\mathcal{N} = 2$ supergravity is reviewed in this volume by Ivanov [52].

调和超空间与投影超空间使用的是同一个超空间 (1)。无需多谈 [46](另见 [45, 47, 48]) 中已阐明的技术细节，二者的区别在于:(i) 所用离壳超多重态的结构不同，(ii) 选取的超对称作用量原理不同。由于这些概念层面的差异，两种方法在许多方面都被证明是互补的。特别是，调和超空间为 $\mathcal{N} = 2$ 超引力 [49, 50] 提供了强有力的预势表述 (综述见 [40])，其核心理念与奥格耶夫斯基-索卡切夫处理 $\mathcal{N} = 1$ 超引力的方法 [51] 类似。已证明投影超空间非常适合为含有 8 个超荷的超引力-物质系统发展协变几何表述。在本文集中，伊万诺夫 [52] 综述了 $\mathcal{N} = 2$ 超引力的调和超空间方法。

The formalism of curved projective superspace was originally developed in 2008 for $\mathcal{N} = 1$ supergravity-matter systems in five dimensions [53,54] using the structure of superconformal projective multiplets [55]. Shortly thereafter, these constructions were generalized to develop the projective-superspace approach for $\mathcal{N} = 2$ matter-coupled supergravity in four dimensions [56-59] (It was subsequently extended to supergravity-matter theories in two [60], three [61], and six [62] dimensions.). With the advent of $\mathcal{N} = 2$ conformal superspace [63], and its applications to component reduction [64], a novel formulation of curved projective superspace has been given in [65, 66]. This approach has also been extended to a novel covariant harmonic superspace framework in [67]. All of these publications followed the philosophy of the $\mathcal{N} = 2$ superconformal tensor calculus to realize supergravity-matter systems as conformal supergravity coupled to superconformal matter multiplets.

弯曲投影超空间的形式体系最初是在 2008 年针对五维 $\mathcal{N} = 1$ 超引力-物质系统，利用超共形投影多重态的结构 [55] 发展出来的 [53,54]。此后不久，这些构造被推广，发展出了四维下 $\mathcal{N} = 2$ 物质耦合超引力的投影超空间方法 [56-59](该方法后来被推广到二维 [60]、三维 [61] 和六维 [62] 的超引力-物质理论)。随着 $\mathcal{N} = 2$ 共形超空间 [63] 出现，并被应用到分量约化 [64]，文献 [65, 66] 给出了一种新的弯曲投影超空间表述。该方法在 [67] 中还被推广到一个新的协变调和超空间框架。所有这些研究都遵循 $\mathcal{N} = 2$ 超共形张量演算的理念，将超引力-物质系统实现为共形超引力耦合超共形物质多重态。

There are three superspace formulations for $\mathcal{N} = 2$ conformal supergravity that have found numerous applications in the recent years, specifically: (i) U (2) superspace [26], (ii) SU(2) superspace [56], and (iii) conformal superspace [63]. The $\mathcal{N} = 2$ conformal superspace of [63] is an ultimate formulation for $\mathcal{N} = 2$ conformal supergravity in the sense that any different off-shell formulation is either equivalent to it or is obtained from it by partially fixing the gauge freedom. In particular, the U(2) and SU(2) superspaces can be derived from conformal superspace by imposing partial gauge fixing conditions (The relationship between the U (2) and SU (2) superspaces is described in [59].). At the component level, $\mathcal{N} = 2$ conformal superspace reduces to the $\mathcal{N} = 2$ superconformal tensor calculus.

近年来，$\mathcal{N} = 2$ 共形超引力共有三种超空间表述，且已得到大量应用，分别是:(i) U (2) 超空间 [26]，(ii) SU(2) 超空间 [56]， 以及 (iii) 共形超空间 [63]。文献 [63] 提出的 $\mathcal{N} = 2$ 共形超空间是 $\mathcal{N} = 2$ 共形超引力的终极表述——任何其他不同的脱壳表述要么等价于它，要么就是通过部分固定规范自由度从它得到的。具体来说，U(2) 与 SU(2) 超空间都可以通过施加部分规范固定条件从共形超空间推导得出 (U (2) 超空间与 SU (2) 超空间的关系已在文献 [59] 中描述)。在分量层面，$\mathcal{N} = 2$ 共形超空间约化为 $\mathcal{N} = 2$ 超共形张量演算。

The $\mathcal{N} = 2$ conformal superspace of [63, 64] is a natural extension of the $\mathcal{N} = 1$ formulation [68]. Conformal superspace approaches have also been developed for extended supergravity-matter systems in three

[69-71], five [72], and six [73,74] dimensions. These references include various applications.

[63, 64] 的 $\mathcal{N} = 2$ 共形超空间是 $\mathcal{N} = 1$ 表述 [68] 的自然推广。共形超空间方法也已被发展，用于三维 [69-71]、五维 [72] 和六维 [73,74] 的扩展超引力-物质系统，这些文献包含了各类应用。

Recently, new supertwistor formulations were discovered for conformal super-gravity theories in diverse dimensions [75]. In the four-dimensional $\mathcal{N} = 2$ case, the supertwistor formulation is expected to be related to conformal superspace; however, relevant technical details have not yet been worked out in the literature.

近年来，不同维度的共形超引力理论都发现了新的超扭矢表述 [75]。对于四维 $\mathcal{N} = 2$ 情形，超扭矢表述被认为和共形超空间相关，但目前文献中还没有给出相关的技术细节。

Our two-component spinor notation and conventions follow [76] and are similar to those adopted in [77]. The only difference is that the spinor Lorentz generators $(\sigma_{ab})_\alpha{}^\beta$ and $(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$ used in [76] have an extra minus sign as compared with [77], specifically $\sigma_{ab} = -\frac{1}{4}(\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a)$ and $\tilde{\sigma}_{ab} = -\frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a)$.

本文的双分量旋量记号与约定遵循文献 [76]，与文献 [77] 采用的约定相近。唯一的区别是，文献 [76] 中使用的旋量洛伦兹生成元 $(\sigma_{ab})_\alpha{}^\beta$ 和 $(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$ 相比文献 [77] 多一个负号，具体为 $\sigma_{ab} = -\frac{1}{4}(\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a)$ 和 $\tilde{\sigma}_{ab} = -\frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a)$ 。

Rigid Superconformal Transformations

刚性超共形变换

We denote by $z^A = \left(x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}}\right)$ the Cartesian coordinates for Minkowski super-space $\mathbb{M}^{4|8}$ and use the notation $D_A = \left(\partial_a, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i\right)$ for the superspace covariant derivatives. The only non-zero graded commutation relation is

我们用 $z^A = \left(x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}}\right)$ 表示闵氏超空间 $\mathbb{M}^{4|8}$ 的笛卡尔坐标，并用记号 $D_A = \left(\partial_a, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i\right)$ 表示超空间协变导数。唯一非零的分次对易关系为

$$\left\{D^i_\alpha, \bar{D}_{\dot{\beta}j}\right\} = -2\mathrm{i}\delta^i_j(\sigma^c)_{\alpha\dot{\beta}}\partial_c = -2\mathrm{i}\delta^i_j\partial_{\alpha\dot{\beta}},\ i, j = \underline{1}, \underline{2}. \tag{5}$$

The $\mathcal{N} = 2$ super-Poincaré algebra has an outer automorphism group $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$, which is also called the R -symmetry group. The $\mathrm{SU}(2)_R$ indices are raised and lowered using the antisymmetric tensor $\varepsilon^{ij} = -\varepsilon^{ji}$ and its inverse ε_{ij} normalized by $\varepsilon^{\underline{12}} = 1$.

$\mathcal{N} = 2$ 超庞加莱代数具有外自同构群 $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ ，该群也被称为 R 对称群。$\mathrm{SU}(2)_R$ 指标利用反对称张量 $\varepsilon^{ij} = -\varepsilon^{ji}$ 及其逆 ε_{ij} 升降，归一化条件为 $\varepsilon^{\underline{12}} = 1$ 。

Conformal Killing Supervector Fields

共形 Killing 超向量场

Superconformal transformations in $\mathbb{M}^{4|8}$ were first studied by Sohnius [78]. Our presentation follows [79].

$\mathbb{M}^{4|8}$ 中的超共形变换最早由 Sohnius 研究 [78]。本文的论述沿用文献 [79] 的内容。

An infinitesimal superconformal transformation $z^A \to z^A + \delta z^A$, with $\delta z^A = \xi z^A = \left(\xi^a + \mathrm{i}\left(\xi_i \sigma^a \bar{\theta}^i - \theta_i \sigma^a \bar{\xi}^i\right), \xi_i^\alpha, \bar{\xi}^i_{\dot\alpha}\right)$, is generated by a conformal Killing supervector field

满足 $\delta z^A = \xi z^A = \left(\xi^a + \mathrm{i}\left(\xi_i \sigma^a \bar{\theta}^i - \theta_i \sigma^a \bar{\xi}^i\right), \xi_i^\alpha, \bar{\xi}^i_{\dot\alpha}\right)$ 的无穷小超共形变换 $z^A \to z^A + \delta z^A$ 由共形 Killing 超向量场生成

$$\xi = \xi^b \partial_b + \xi^\beta_j D^j_\beta + \bar{\xi}^j_{\dot\beta} \bar{D}^{\dot\beta}_j = \bar{\xi}. \tag{6}$$

The defining property of ξ is

ξ 的定义性质为

$$\left[\xi, D^i_\alpha\right] = -\left(D^i_\alpha \xi^\beta_j\right) D^j_\beta. \tag{7}$$

This condition implies the relations

该条件可推导出关系

$$\bar{D}^{\dot\alpha}_i \xi^\beta_j = 0,\ \bar{D}^{\dot\alpha}_i \xi^{\dot\beta\beta} = 4\mathrm{i}\varepsilon^{\dot\alpha\dot\beta} \xi^\beta_i \ \Rightarrow\ \xi^\alpha_i = -\frac{\mathrm{i}}{8} \bar{D}_{\dot\alpha i} \xi^{\dot\alpha\alpha} \tag{8}$$

and their complex conjugates, and therefore

以及它们的复共轭，因此可得

$$\bar{D}_{(\alpha i} \xi_{\beta)\dot\beta} = 0,\ \bar{D}^i_{(\dot\alpha} \xi_{\beta\dot\beta)} = 0 \Rightarrow\ \partial_{(\alpha(\dot\alpha} \xi_{\beta)\dot\beta)} = 0. \tag{9}$$

It then follows that

进而可得

$$\left[\xi, D^i_\alpha\right] = -K_\alpha{}^\beta\left[\xi\right] D^i_\beta - \frac{1}{2}\bar{\sigma}\left[\xi\right] D^i_\alpha - \Lambda^i{}_j\left[\xi\right] D^j_\alpha. \tag{10}$$

Here we have introduced the chiral Lorentz $K_{\beta\gamma}\left[\xi\right]$ and super-Weyl $\sigma\left[\xi\right]$ parameters, as well as the $\mathrm{SU}(2)_R$ parameter $\Lambda^{ij}\left[\xi\right]$ defined by

此处我们引入了手征洛伦兹参数 $K_{\beta\gamma}[\xi]$ 和超外尔参数 $\sigma[\xi]$，以及由下式定义的 $\mathrm{SU}(2)_R$ 参数 $\Lambda^{ij}[\xi]$

$$K_{\alpha\beta}[\xi]=\frac{1}{2}D^i_{(\alpha}\xi_{\beta)i}=K_{\beta\alpha}[\xi]\,,\ \bar{D}^{\dot\alpha}_i K_{\alpha\beta}[\xi]=0, \tag{11a}$$

$$\sigma[\xi]=\frac{1}{2}\bar{D}^{\dot\alpha}_i\overline{\xi}^i_{\dot\alpha},\ \bar{D}^{\dot\alpha}_i\sigma[\xi]=0, \tag{11b}$$

$$\Lambda^{ij}[\xi]=-\frac{\mathrm{i}}{16}\left[D^{(i}_\alpha,\bar{D}^{j)}_{\dot\alpha}\right]\xi^{\alpha\dot\alpha}=\Lambda^{ji}[\xi]\,,\ \overline{\Lambda^{ij}[\xi]}=\Lambda_{ij}[\xi]\,. \tag{11c}$$

We recall that the Lorentz parameters with vector and spinor indices are related to each other as follows: $K^{bc}[\xi]=\left(\sigma^{bc}\right)_{\beta\gamma}K^{\beta\gamma}[\xi]-\left(\tilde\sigma^{bc}\right)_{\dot\beta\dot\gamma}\bar{K}^{\dot\beta\dot\gamma}[\xi]$. The parameters in (11) obey several first-order differential properties:

我们回顾: 带矢量指标和旋量指标的洛伦兹参数满足如下关系: $K^{bc}[\xi]=\left(\sigma^{bc}\right)_{\beta\gamma}K^{\beta\gamma}[\xi]-\left(\tilde\sigma^{bc}\right)_{\dot\beta\dot\gamma}\bar{K}^{\dot\beta\dot\gamma}[\xi]$。式 (11) 中的参数满足多个一阶微分性质:

$$D^i_\alpha\Lambda^{jk}[\xi]=\varepsilon^{i(j}D^{k)}_\alpha\sigma[\xi] \tag{12a}$$

$$D^i_\alpha K_{\beta\gamma}[\xi]=-\varepsilon_{\alpha(\beta}D^i_{\gamma)}\sigma[\xi]\,, \tag{12b}$$

and therefore

因此

$$D^{(i}_\alpha\Lambda^{jk)}[\xi]=\bar{D}^{(i}_{\dot\alpha}\Lambda^{jk)}[\xi]=0, \tag{13a}$$

$$D^i_\alpha D^j_\beta\sigma[\xi]=0. \tag{13b}$$

The most general conformal Killing supervector field has the form

最一般的共形 Killing 超向量场具有如下形式

$$\xi^{\dot\alpha\alpha}_+=a^{\dot\alpha\alpha}+\frac{1}{2}\left(\sigma+\overline{\sigma}\right)y^{\dot\alpha\alpha}+\bar{K}^{\dot\alpha}{}_{\dot\beta}y^{\dot\beta\alpha}+y^{\dot\alpha\beta}K_\beta{}^\alpha-y^{\dot\alpha\beta}b_{\beta\dot\beta}y^{\dot\beta\alpha}$$

$$+4\mathrm{i}\overline{\varepsilon}^{\dot\alpha i}\theta^\alpha_i-4y^{\dot\alpha\beta}\eta^i_\beta\theta^\alpha_i, \tag{14a}$$

$$\xi^\alpha_i=\varepsilon^\alpha_i+\frac{1}{2}\overline{\sigma}\theta^\alpha_i+\theta^\beta_i K_\beta{}^\alpha+\Lambda_i{}^j\theta^\alpha_j-\theta^\beta_i b_{\beta\dot\beta}y^{\dot\beta\alpha}$$

$$-\mathrm{i}\overline{\eta}_{\dot\beta i}y^{\dot\beta\alpha}-4\theta^\alpha_i\eta^j_\beta\theta^\beta_j, \tag{14b}$$

where we have introduced the complex four-vector

其中我们引入了复四矢量

$$\xi_+^a = \xi^a + 2i\xi_i\sigma^a\overline{\theta}^i, \ \overline{\xi}^a = \xi^a, \tag{15}$$

along with the complex bosonic coordinates $y^a = x^a + i\theta_i\sigma^a\overline{\theta}^i$ of the chiral subspace of $\mathbb{M}^{4|8}$. The constant bosonic parameters in (14) correspond to the spacetime translation $\left(a^{\dot{\alpha}\alpha}\right)$, Lorentz transformation $\left(K_\beta{}^\alpha, \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}}\right)$, SU $(2)_R$ transformation $\left(\Lambda^{ij} = \Lambda^{ji}\right)$, special conformal transformation $\left(b_{\alpha\dot{\beta}}\right)$, and combined scale and $\mathrm{U}(1)_R$ transformations $(\sigma = \tau - 2i\varphi)$. The constant fermionic parameters in (14) correspond to the Q -supersymmetry $\left(\varepsilon_i^\alpha\right)$ and S -supersymmetry $\left(\eta_i^\alpha\right)$ transformations. The constant parameters $K_{\alpha\beta}, \Lambda^{ij}$, and σ are obtained from $K_{\alpha\beta}[\xi]$, $\Lambda^{ij}[\xi]$, and $\sigma[\xi]$, respectively, by setting $z^A = 0$.

以及 $\mathbb{M}^{4|8}$ 的手性子空间的复玻色坐标 $y^a = x^a + i\theta_i\sigma^a\overline{\theta}^i$ 。式 (14) 中的常数玻色参数分别对应时空平移 $\left(a^{\dot{\alpha}\alpha}\right)$ 、洛伦兹变换 $\left(K_\beta{}^\alpha, \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}}\right)$, SU $(2)_R$ 、伸缩变换 $\left(\Lambda^{ij} = \Lambda^{ji}\right)$ 、特殊共形变换 $\left(b_{\alpha\dot{\beta}}\right)$ ， 以及标度与 $\mathrm{U}(1)_R$ 的组合变换 $(\sigma = \tau - 2i\varphi)$ 。式 (14) 中的常数费米参数分别对应 Q 超对称变换 $\left(\varepsilon_i^\alpha\right)$ 和 S 超对称变换 $\left(\eta_i^\alpha\right)$ 。令 $z^A = 0$ 即可分别从 $K_{\alpha\beta}[\xi]$, $\Lambda^{ij}[\xi]$ 和 $\sigma[\xi]$ 得到常数参数 $K_{\alpha\beta}, \Lambda^{ij}$ 和 σ 。

It is useful to introduce a condensed notation for the superconformal parameters

为超共形参数引入缩记法是很方便的

$$\lambda^{\tilde{a}} = \left(a^A, K^{ab}, \Lambda^{ij}, \tau, \varphi, b_A\right), \ a^A := \left(a^a, \varepsilon_i^\alpha, \overline{\varepsilon}_{\dot{\alpha}}^i\right), \ b_A := \left(b_a, \eta_\alpha^i, \overline{\eta}_i^{\dot{\alpha}}\right), \tag{16}$$

as well as for the generators of the superconformal group

同样也可以为超共形群的生成元引入缩记法

$$X_{\tilde{a}} = \left(P_A, M_{ab}, J_{ij}, \mathbb{D}, \mathbb{Y}, K^A\right), \ P_A := \left(P_a, Q_\alpha^i, \bar{Q}_i^{\dot{\alpha}}\right), \ K^A := \left(K^a, S_i^\alpha, \bar{S}_{\dot{\alpha}}^i\right).$$

(17)

The general conformal Killing supervector field on $\mathbb{C}^{4|2}$,

$\mathbb{C}^{4|2}$ 上的一般共形 Killing 超向量场,

$$\xi = \xi_+^a(y,\theta)\frac{\partial}{\partial y^a} + \xi_i^\alpha(y,\theta)\frac{\partial}{\partial\theta_i^\alpha} \equiv \xi_+^a\partial/\partial y^a + \xi_i^\alpha\partial_\alpha^i, \tag{18}$$

may be written in the form:

可以写成如下形式:

$$\xi = \lambda^{\tilde{a}}\xi_{\tilde{a}}(X) = a^A\xi_A(P) + \frac{1}{2}K^{ab}\xi_{ab}(M) + \Lambda^{ij}\xi_{ij}(J) + \tau\xi(\mathbb{D})$$

$$+ i\varphi\xi(\mathbb{Y}) + b_A\xi^A(K). \tag{19}$$

We read off the relevant supervector fields:

我们可以直接读出相关的超向量场:

$$\xi_a(P) = \partial/\partial y^a,\ \xi^i_\alpha(P) = \partial^i_\alpha,\ \bar{\xi}^{\dot\alpha}_i(P) = -2\mathrm{i}(\tilde\sigma^c\theta_i)^{\dot\alpha}\partial/\partial y^c, \tag{20a}$$

$$\xi_{ab}(M) = y_a\partial/\partial y^b - y_b\partial/\partial y^a + (\theta_i\sigma_{ab})^\gamma\partial^i_\gamma,\ \xi_{ij}(J) = \theta^\alpha_{(i}\partial_{\alpha j)}, \tag{20b}$$

$$\xi(\mathbb{D}) = y^c\partial/\partial y^c + \frac{1}{2}\theta^\gamma_i\partial^i_\gamma,\ \xi(\mathbb{Y}) = \theta^\gamma_i\partial^i_\gamma, \tag{20c}$$

$$\xi^a(K) = 2y^a y^c\partial/\partial y^c - y^2\partial/\partial y^a - (\theta_i\sigma^a\tilde\sigma^c)^\gamma y_c\partial^i_\gamma, \tag{20d}$$

$$\xi^\alpha_i(K) = 2\left(\theta\sigma^c\tilde\sigma^d\right)^\alpha y_d\partial/\partial y^c + 4\theta^\alpha_i\theta^\beta_j\partial^j_\beta, \tag{20e}$$

$$\bar{\xi}^i_{\dot\alpha}(K) = \mathrm{i}(\sigma^c)^\gamma{}_{\dot\alpha}y_c\partial^i_\gamma. \tag{20f}$$

Making use of the above operators, the graded commutation relations for the superconformal algebra, $\left[X_{\tilde a}, X_{\tilde b}\right\} = -f_{\tilde a\tilde b}{}^{\tilde c}X_{\tilde c}$, can be derived keeping in mind the relation

利用上述算子，结合记住该关系，可以推导出超共形代数 $\left[X_{\tilde a}, X_{\tilde b}\right\} = -f_{\tilde a\tilde b}{}^{\tilde c}X_{\tilde c}$ 的分次对易关系

$$\xi = \lambda^{\tilde a}\xi_{\tilde a}(X) \to \delta_\xi = \lambda^{\tilde a}X_{\tilde a},\ [\xi_1, \xi_2] \to -\left[\delta_{\xi_1}, \delta_{\xi_2}\right]. \tag{21}$$

Superconformal Algebra

超共形代数

Here we describe the graded commutation relations for the $\mathcal{N} = 2$ superconformal algebra $\mathfrak{su}(2,2\mid 2)$. We start with the commutation relations for the conformal algebra:

我们在这里描述 $\mathcal{N} = 2$ 超共形代数 $\mathfrak{su}(2,2\mid 2)$ 的分次对易关系。我们从共形代数的对易关系开始:

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c}, \tag{22a}$$

$$[M_{ab}, P_c] = 2\eta_{c[a}P_{b]}, \tag{22b}$$

$$[M_{ab}, K_c] = 2\eta_{c[a}K_{b]}, \tag{22c}$$

$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab}. \tag{22d}$$

The R -symmetry generators $\mathbb{Y}$ and J_{ij} commute with all the generators of the conformal group. Amongst themselves, they obey the algebra:

R 对称生成元 $\mathbb{Y}$ 和 J_{ij} 与共形群的所有生成元对易。它们自身满足该代数:

$$\left[J^{ij}, J^{kl}\right] = \varepsilon^{k(i}J^{j)l} + \varepsilon^{l(i}J^{j)k}. \tag{23}$$

The superconformal algebra is then obtained by extending the translation generator to P_A and the special conformal generator to K^A . The commutation relations involving the Q -supersymmetry generators with the bosonic ones are:

超共形代数可以通过将平移生成元延拓到 P_A 、将特殊共形生成元延拓到 K^A 得到。涉及 Q 超对称生成元与玻色生成元的对易关系为:

$$\left[M_{ab}, Q^i_\gamma\right] = (\sigma_{ab})_\gamma{}^\delta Q^i_\delta,\ \left[M_{ab}, \bar{Q}^{\dot\gamma}_i\right] = (\tilde\sigma_{ab})^{\dot\gamma}{}_{\dot\delta}\bar{Q}^{\dot\delta}_i, \tag{24a}$$

$$\left[\mathbb{D}, Q^i_\alpha\right] = \frac{1}{2}Q^i_\alpha,\ \left[\mathbb{D}, \bar{Q}^{\dot\alpha}_i\right] = \frac{1}{2}\bar{Q}^{\dot\alpha}_i, \tag{24b}$$

$$\left[\mathbb{Y}, Q^i_\alpha\right] = Q^i_\alpha,\ \left[\mathbb{Y}, \bar{Q}^{\dot\alpha}_i\right] = -\bar{Q}^{\dot\alpha}_i, \tag{24c}$$

$$\left[J_{ij}, Q^k_\alpha\right] = -\delta^k_{(i}Q_{\alpha j)},\ \left[J_{ij}, \bar{Q}^{\dot\alpha}_k\right] = -\varepsilon_{k(i}\bar{Q}^{\dot\alpha}_{j)}, \tag{24d}$$

$$\left[K^a, Q^i_\beta\right] = -\mathrm{i}(\sigma^a)_\beta{}^{\dot\beta}\bar{S}^i_{\dot\beta},\ \left[K^a, \bar{Q}^{\dot\beta}_i\right] = -\mathrm{i}(\sigma^a)^{\dot\beta}{}_\beta S^\beta_i. \tag{24e}$$

The commutation relations involving the S -supersymmetry generators with the bosonic operators are:

涉及 S 超对称生成元与玻色算符的对易关系为:

$$\left[M_{ab}, S^\gamma_i\right] = -(\sigma_{ab})_\beta{}^\gamma S^\beta_i,\ \left[M_{ab}, \bar{S}^i_{\dot\gamma}\right] = -(\tilde\sigma_{ab})^{\dot\beta}{}_{\dot\gamma}\bar{S}^i_{\dot\beta}, \tag{25a}$$

$$\left[\mathbb{D}, S^\alpha_i\right] = -\frac{1}{2}S^\alpha_i,\ \left[\mathbb{D}, \bar{S}^i_{\dot\alpha}\right] = -\frac{1}{2}\bar{S}^i_{\dot\alpha}, \tag{25b}$$

$$\left[\mathbb{Y}, S^\alpha_i\right] = -S^\alpha_i,\ \left[\mathbb{Y}, \bar{S}^i_{\dot\alpha}\right] = \bar{S}^i_{\dot\alpha}, \tag{25c}$$

$$\left[J_{ij}, S^\alpha_k\right] = -\varepsilon_{k(i}S^\alpha_{j)},\ \left[J_{ij}, \bar{S}^k_{\dot\alpha}\right] = -\delta^k_{(i}\bar{S}_{\dot\alpha j)}, \tag{25d}$$

$$\left[S^\alpha_i, P_b\right] = \mathrm{i}(\sigma_b)^\alpha{}_{\dot\beta}\bar{Q}^{\dot\beta}_i,\ \left[\bar{S}^i_{\dot\alpha}, P_b\right] = \mathrm{i}(\sigma_b)_{\dot\alpha}{}^\beta Q^i_\beta. \tag{25e}$$

Finally, the anti-commutation relations of the fermionic generators are:

最后，费米子生成元的反对易关系为:

$$\left\{Q^i_\alpha, \bar{Q}^{\dot{\alpha}}_j\right\} = -2\mathrm{i}\delta^i_j\left(\sigma^b\right)_\alpha{}^{\dot{\alpha}}P_b = -2\mathrm{i}\delta^i_jP_\alpha{}^{\dot{\alpha}}, \tag{26a}$$

$$\left\{S^\alpha_i, \bar{S}^j_{\dot{\alpha}}\right\} = 2\mathrm{i}\delta^j_i\left(\sigma^b\right)^\alpha{}_{\dot{\alpha}}K_b = 2\mathrm{i}\delta^j_iK^\alpha{}_{\dot{\alpha}}, \tag{26b}$$

$$\left\{S^\alpha_i, Q^j_\beta\right\} = \delta^j_i\delta^\alpha_\beta\left(2\mathbb{D} - \mathbb{Y}\right) - 4\delta^j_iM^\alpha{}_\beta + 4\delta^\alpha_\beta J_i{}^j, \tag{26c}$$

$$\left\{\bar{S}^i_{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}_j\right\} = \delta^i_j\delta^{\dot{\beta}}_{\dot{\alpha}}\left(2\mathbb{D} + \mathbb{Y}\right) + 4\delta^i_j\bar{M}_{\dot{\alpha}}{}^{\dot{\beta}} - 4\delta^{\dot{\beta}}_{\dot{\alpha}}J^i{}_j, \tag{26d}$$

where $M_{\alpha\beta} = \frac{1}{2}\left(\sigma^{ab}\right)_{\alpha\beta}M_{ab}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\left(\tilde{\sigma}^{ab}\right)_{\dot{\alpha}\dot{\beta}}M_{ab}$. Note that all remaining (anti-)commutators not explicitly listed above vanish identically.

其中 $M_{\alpha\beta} = \frac{1}{2}\left(\sigma^{ab}\right)_{\alpha\beta}M_{ab}$ 和 $\bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\left(\tilde{\sigma}^{ab}\right)_{\dot{\alpha}\dot{\beta}}M_{ab}$ 。注意所有上述未明确列出的其余 (反) 对易子都恒等于零。

The graded commutation relations (22)-(26) constitute the $\mathcal{N} = 2$ superconfor-mal algebra, $\mathfrak{su}\left(2,2 \mid 2\right)$. Its generators obey the graded Jacobi identity

分次对易关系 (22)-(26) 构成了 $\mathcal{N} = 2$ 超共形代数 $\mathfrak{su}\left(2,2 \mid 2\right)$ 。它的生成元满足分次雅可比恒等式

$$(-1)^{\varepsilon_{\tilde{a}}\varepsilon_{\tilde{c}}}\left[X_{\tilde{a}}, \left[X_{\tilde{b}}, X_{\tilde{c}}\right\}\right\} + \text{ (two cycles) } = 0 \tag{27}$$

where $\varepsilon_{\tilde{a}} = \varepsilon\left(X_{\tilde{a}}\right)$ is the Grassmann parity of the generator $X_{\tilde{a}}$. Making use of $\left[X_{\tilde{a}}, X_{\tilde{b}}\right\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$, the Jacobi identities are equivalently written as

其中 $\varepsilon_{\tilde{a}} = \varepsilon\left(X_{\tilde{a}}\right)$ 是生成元 $X_{\tilde{a}}$ 的格拉斯曼奇偶性。利用 $\left[X_{\tilde{a}}, X_{\tilde{b}}\right\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$ ， 雅可比恒等式可以等价写为

$$f_{[\tilde{a}\tilde{b}}{}^{\tilde{d}}f_{|\tilde{d}|\tilde{c}\}}{}^{\tilde{e}} = 0. \tag{28}$$

Superconformal Primary Multiplets

超共形主多重态

The superconformal transformation law of a primary tensor superfield (with suppressed indices) is

(指标隐去的) 主张量超场的超共形变换规律为

$$\delta_\xi U = \mathcal{K}\left[\xi\right]U$$

$$\mathcal{K}\left[\xi\right] = \xi + \frac{1}{2}K^{ab}\left[\xi\right]M_{ab} + \Lambda^{ij}\left[\xi\right]J_{ij} + p\sigma\left[\xi\right] + q\overline{\sigma}\left[\xi\right]. \tag{29}$$

Here the generators M_{ab} and J_{ij} act on the Lorentz and SU(2) indices of U , respectively. The parameters p and q are related to the dimension (or Weyl weight) w and $\mathrm{U}(1)_R$ charge c of U as $p+q=w$ and $p-q=-\frac{1}{2}c$.

此处生成元 M_{ab} 和 J_{ij} 分别作用在 U 的洛伦兹指标与 SU(2) 指标上。参数 p 和 q 与 U 的维度 (或外尔权) w 和 $\mathrm{U}(1)_R$ 荷 c 满足关系 $p+q=w$ 和 $p-q=-\frac{1}{2}c$ 。

As an example, let us consider a chiral tensor superfield $\phi, \bar{D}_i^{\dot{\alpha}}\phi=0$. Requiring it to be primary leads to the conditions

举例来说，我们考虑手征张量超场 $\phi, \bar{D}_i^{\dot{\alpha}}\phi=0$ 。要求它为主场可得条件

$$\bar{M}_{\dot{\alpha}\dot{\beta}}\phi=0,\ J_{ij}\phi=0,\ q=0. \tag{30}$$

These conditions imply that (i) ϕ can carry only undotted spinor indices,(ii) ϕ must be neutral under the group $\mathrm{SU}(2)_R$, and (iii) the dimension w and the $\mathrm{U}(1)_R$ charge c of ϕ are related as $c=-2w$. A chiral scalar W is called reduced if it obeys the reality condition

这些条件表明:(i) ϕ 只能携带无点旋量指标，(ii) ϕ 在群 $\mathrm{SU}(2)_R$ 下必须是中性的，(iii) ϕ 的维度 w 和 $\mathrm{U}(1)_R$ 荷 c 满足关系 $c=-2w$ 。若手征标量 W 满足实条件，则称之为约化手征标量

$$D^{ij}W=\bar{D}^{ij}\bar{W},\ D^{ij}\ :=\ D^{\alpha(i}D_{\alpha}^{j)},\ \bar{D}_{ij}\ :=\ \bar{D}_{\dot{\alpha}(i}\bar{D}_{j)}^{\dot{\alpha}}, \tag{31}$$

which uniquely fixes the dimension of W to be $+1$. Chiral multiplets exist both in the $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric cases. New types of multiplets are offered by $\mathcal{N}=2$ supersymmetry, as will be discussed below.

该条件将 W 的维度唯一确定为 $+1$。手征多重态既存在于 $\mathcal{N}=1$ 超对称情形，也存在于 $\mathcal{N}=2$ 超对称情形。如下文将要讨论的，$\mathcal{N}=2$ 超对称会给出新型多重态。

An $\mathcal{O}(n)$ multiplet $H^{i_1\cdots i_n}=H^{(i_1\cdots i_n)}$ obeys the analyticity constraints (The $\mathcal{O}(n)$ multiplets are well-known in the literature on $\mathcal{N}=2$ supersymmetric field theories. The $n=1$ case corresponds to the on-shell Fayet-Sohnius hypermultiplet [30, 80] . The field strength of the $\mathcal{N}=2$ tensor multiplet [81] is described by a real

一个 $\mathcal{O}(n)$ 多重态 $H^{i_1\cdots i_n}=H^{(i_1\cdots i_n)}$ 满足解析性约束 ($\mathcal{O}(n)$ 多重态在 $\mathcal{N}=2$ 超对称场论的文献中已有广泛研究。$n=1$ 情形对应在壳 Fayet-Sohnius 超多重态 [30, 80] 。$\mathcal{N}=2$ 张量多重态 [81] 的场强由一个实

$\mathcal{O}(2)$ multiplet [20, 24, 82] . General $\mathcal{O}(n)$ multiplets, with $n>2$, were introduced in [33, 42, 83] . The case $n=4$ was first studied in [82].)

$\mathcal{O}(2)$ 多重态 [20, 24, 82] 描述。满足 $n>2$ 的一般 $\mathcal{O}(n)$ 多重态由文献 [33, 42, 83] 引入，$n=4$ 情形最早在文献 [82] 中研究。)

$$D_\alpha^{(i_1} H^{i_2 \cdots i_{n+1})} = 0,\ \bar{D}_{\dot\alpha}^{(i_1} H^{i_2 \cdots i_{n+1})} = 0. \tag{32}$$

In the super-Poincaré case, these constraints are consistent with $H^{i_1 \cdots i_n}$ carrying Lorentz indices in addition to the SU(2) ones. However, this is no longer allowed if $H^{i_1 \cdots i_n}$ is a superconformal primary multiplet. Then, the superconformal transformation law of H is uniquely determined by the constraints (32) to be

在超庞加莱情形下，这些约束与 $H^{i_1 \cdots i_n}$ 除 SU(2) 指标外还携带洛伦兹指标是相容的。但若 $H^{i_1 \cdots i_n}$ 是超共形主多重态，该情况就不再被允许。此时 H 的超共形变换规律可由约束 (32) 唯一确定为

$$\delta_\xi H^{i_1 \cdots i_n} = \xi H^{i_1 \cdots i_n} + n \Lambda_j{}^{(i_1[\xi])} H^{i_2 \cdots i_n j} + \frac{n}{2} \left(\sigma[\xi] + \bar\sigma[\xi] \right) H^{i_1 \cdots i_n}. \tag{33}$$

In the case that n is even, $n = 2m$, this transformation law is compatible with the reality condition $\overline{H^{i_1 \dots i_{2m}}} = H_{i_1 \dots i_{2m}} = \varepsilon_{i_1 j_1} \dots \varepsilon_{i_{2m} j_{2m}} H^{j_1 \dots j_{2m}}$.

当 n 为偶，即 $n = 2m$ 时，该变换规律与实条件 $\overline{H^{i_1 \dots i_{2m}}} = H_{i_1 \dots i_{2m}} = \varepsilon_{i_1 j_1} \dots \varepsilon_{i_{2m} j_{2m}} H^{j_1 \dots j_{2m}}$ 相容。

Superconformal Projective Multiplets

超共形投影多重态

The constraints (32) can be given a more transparent interpretation if one makes use of an isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$ that allows one to introduce new spinor covariant derivatives,

如果利用同扭量 $v^i \in \mathbb{C}^2 \setminus \{0\}$ 引入新的旋量协变导数，约束条件 (32) 就可以得到更清晰的解释，

$$D_\alpha^{(1)} = v_i D_\alpha^i,\ \bar{D}_{\dot\alpha}^{(1)} = v_i \bar{D}_{\dot\alpha}^i,\ v_i := \varepsilon_{ij} v^j, \tag{34}$$

and to convert $H^{i_1 \cdots i_n}(z)$ into an index-free homogeneous polynomial of degree n ,

并且可以将 $H^{i_1 \cdots i_n}(z)$ 转化为一个次数为 n 的无指标齐次多项式，

$$H^{(n)}(z, v) = v_{i_1} \cdots v_{i_n} H^{i_1 \cdots i_n}(z). \tag{35}$$

In accordance with (5), the operators (34) strictly anticommute with each other,

根据式 (5)，算符 (34) 两两严格反对易，

$$\left\{ D_\alpha^{(1)}, D_\beta^{(1)} \right\} = \left\{ \bar{D}_{\dot\alpha}^{(1)}, \bar{D}_{\dot\beta}^{(1)} \right\} = \left\{ D_\alpha^{(1)}, \bar{D}_{\dot\beta}^{(1)} \right\} = 0, \tag{36}$$

and annihilate $H^{(n)}$,

并且零化 $H^{(n)}$ ，

$$D_\alpha^{(1)} H^{(n)} = 0,\ \bar{D}_{\dot{\alpha}}^{(1)} H^{(n)} = 0. \tag{37}$$

These constraints do not change if we replace $v^i \to \mathfrak{c} v^i$, with $\mathfrak{c} \in \mathbb{C} \setminus \{0\} \equiv \mathbb{C}^*$, in the definition of the operators (34) and the superfield (35). We see that the isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$ is defined modulo the equivalence relation $v^i \sim \mathfrak{c} v^i$, with $\mathfrak{c} \in \mathbb{C}^*$; hence, it provides homogeneous coordinates for $\mathbb{C}P^1$. The superfield (35) can be interpreted to be a holomorphic tensor field on the superspace (1).

在算符 (34) 和超场 (35) 的定义中，若将 $v^i \to \mathfrak{c} v^i$ 替换为 $\mathfrak{c} \in \mathbb{C} \setminus \{0\} \equiv \mathbb{C}^*$ ，这些约束保持不变。可以看到同扭量 $v^i \in \mathbb{C}^2 \setminus \{0\}$ 是模等价关系 $v^i \sim \mathfrak{c} v^i$ 定义的，其中 $\mathfrak{c} \in \mathbb{C}^*$ ；因此它给出了 $\mathbb{C}P^1$ 的齐次坐标。超场 (35) 可以解释为超空间 (1) 上的全纯张量场。

The superconformal transformation law (33) can be recast in terms of $H^{(n)}$. For this it is useful to introduce a new isotwistor u^i such that v^i and u^i form a basis for $\mathbb{C}^2$, that is, $(v, u) := v^i u_i \neq 0$.

超共形变换律 (33) 可以用 $H^{(n)}$ 重新表述。为此引入新的同扭量 u^i 是很方便的，此时 v^i 和 u^i 构成 $\mathbb{C}^2$ 的一组基，即满足 $(v, u) := v^i u_i \neq 0$ 。

$$\delta_\xi H^{(n)} = \left(\xi - \Lambda^{(2)}[\xi]\, \partial^{(-2)}\right) H^{(n)} + n \sum [\xi]\, H^{(n)}. \tag{38}$$

Here we have introduced the differential operator

这里我们引入了微分算符

$$\partial^{(-2)} := \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i}, \tag{39}$$

as well as the parameters

以及参数

$$\Lambda^{(2)}[\xi] := v_i v_j \Lambda^{ij}[\xi],\ \Lambda^{(0)}[\xi] := \frac{v_i u_j}{(v, u)} \Lambda^{ij}[\xi], \tag{40a}$$

$$\sum [\xi] := \Lambda^{(0)}[\xi] + \frac{1}{2}\left(\sigma[\xi] + \bar{\sigma}[\xi]\right), \tag{40b}$$

which have the following properties:

它们具有如下性质:

$$D_\alpha^{(1)} \Lambda^{(2)}[\xi] = 0,\ \bar{D}_{\dot{\alpha}}^{(1)} \Lambda^{(2)}[\xi] = 0, \tag{41a}$$

$$D_\alpha^{(1)} \sum [\xi] = 0,\ \bar{D}_{\dot{\alpha}}^{(1)} \sum [\xi] = 0. \tag{41b}$$

The variation (38) obeys the analyticity constraints (37) due to the identity

变分 (38) 由于下述恒等式满足解析性约束 (37)

$$\left[\xi - \Lambda^{(2)}[\xi]\,\partial^{(-2)}, D_\alpha^{(1)}\right] = -K_\alpha{}^\beta[\xi]\,D_\beta^{(1)} - \left(\frac{1}{2}\sigma[\xi] + \Lambda^{(0)}[\xi]\right)D_\alpha^{(1)}, \tag{42}$$

and a similar relation for $\bar{D}_{\dot\alpha}^{(1)}$.

以及对 $\bar{D}_{\dot\alpha}^{(1)}$ 的一个类似关系。

The above discussion can be extended to more general superconformal projective multiplets [55,84]. A superconformal projective multiplet of weight $n, Q^{(n)}(z,v)$, is a superfield that lives on $\mathbb{R}^{4|8}$ with respect to the superspace variables z^A , is holomorphic with respect to the isotwistor variables v^i on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

上述讨论可以推广到更一般的超共形投影多重态 [55,84]。权重为 $n, Q^{(n)}(z,v)$ 的超共形投影多重态是这样一个超场: 它关于超空间变量 z^A 定义在 $\mathbb{R}^{4|8}$ 上，在 $\mathbb{C}^2 \setminus \{0\}$ 的开区域上关于同扭量变量 v^i 是全纯的，并且由下述条件刻画:

(a) It obeys the analyticity constraints

(a) 它满足解析性约束

$$D_\alpha^{(1)}Q^{(n)} = 0,\ \bar{D}_{\dot\alpha}^{(1)}Q^{(n)} = 0; \tag{43a}$$

(b) It is a homogeneous function of v of degree n ,

(b) 它是 v 的 n 次齐次函数,

$$Q^{(n)}(z, \mathfrak{c}v) = \mathfrak{c}^n Q^{(n)}(z,v),\ \ \mathfrak{c} \in \mathbb{C}^*; \tag{43b}$$

(c) It possesses the superconformal transformation law,

(c) 它具有超共形变换律,

$$\delta_\xi Q^{(n)} = \left(\xi - \Lambda^{(2)}[\xi]\,\partial^{(-2)}\right)Q^{(n)} + n\sum[\xi]\,Q^{(n)}. \tag{43c}$$

By construction, the superfield $Q^{(n)}$ is independent of the isotwistor u^i ,

根据构造，超场 $Q^{(n)}$ 不依赖于同位扭量 u^i ,

$$\partial^{(2)}Q^{(n)} = 0,\ \partial^{(2)}\ :=\ (v,u)\,v^i\frac{\partial}{\partial u^i}. \tag{44}$$

One may check that the variation $\delta_\xi Q^{(n)}$, Eq. (43c), is characterized by the same property, $\partial^{(2)}\delta_\xi Q^{(n)} = 0$, due to the homogeneity condition (43b).

可以验证，式 (43c) 中的变分 $\delta_\xi Q^{(n)}$ 因齐次条件 (43b)，也具有相同性质 $\partial^{(2)}\delta_\xi Q^{(n)} = 0$, There exists a real structure on the space of projective multiplets [35, 36, 42] . Given a weight- n projective multiplet $Q^{(n)}\left(v^i\right)$, its smile conjugate $\breve{Q}^{(n)}\left(v^i\right)$ is defined by

投影多重态空间 [35, 36, 42] 上存在实结构。给定一个权为 n 的投影多重态 $Q^{(n)}\left(v^i\right)$，其微笑共轭 $\breve{Q}^{(n)}\left(v^i\right)$ 定义为:

$$Q^{(n)}\left(v^i\right) \to \bar{Q}^{(n)}\left(\bar{v}_i\right) \to \bar{Q}^{(n)}\left(\bar{v}_i \to -v_i\right) =: \breve{Q}^{(n)}\left(v^i\right), \tag{45}$$

with $\bar{Q}^{(n)}\left(\bar{v}_i\right) := \overline{Q^{(n)}\left(v^i\right)}$ the complex conjugate of $Q^{(n)}\left(v^i\right)$ and $\bar{v}_i$ the complex conjugate of v^i . One can show that $Q^{(n)}(v)$ is a weight- n projective multiplet. In particular, $\breve{Q}^{(n)}(v)$ obeys the analyticity constraints $D^{(1)}_\alpha \breve{Q}^{(n)} = 0$ and $\bar{D}^{(1)}_{\dot\alpha} \breve{Q}^{(n)} = 0$, unlike the complex conjugate of $Q^{(n)}(v)$. One can also check that

其中 $\bar{Q}^{(n)}\left(\bar{v}_i\right) := \overline{Q^{(n)}\left(v^i\right)}$ 是 $Q^{(n)}\left(v^i\right)$ 的复共轭，$\bar{v}_i$ 是 v^i 的复共轭。可以证明 $Q^{(n)}(v)$ 是一个权为 n 的投影多重态。特别地，$\breve{Q}^{(n)}(v)$ 满足解析性约束 $D^{(1)}_\alpha \breve{Q}^{(n)} = 0$ 和 $\bar{D}^{(1)}_{\dot\alpha} \breve{Q}^{(n)} = =0$，这与 $Q^{(n)}(v)$ 的复共轭不同。还可以验证

$$\breve{Q}^{(n)}(v) = (-1)^n Q^{(n)}(v). \tag{46}$$

Therefore, if n is even, one can define real projective multiplets, which are constrained by $Q^{(2n)} = Q^{(2n)}$. Note that geometrically, the smile-conjugation is complex conjugation composed with the antipodal map on the projective space $\mathbb{C}P^1$.

因此，若 n 为偶数，则可以定义实投影多重态，这类多重态受 $Q^{(2n)} = Q^{(2n)}$ 约束。注意，从几何上看，微笑共轭是复共轭复合投影空间 $\mathbb{C}P^1$ 上对径映射得到的变换。

The $\mathcal{O}(n)$ multiplets, $H^{(n)}(v)$, are well defined on the entire projective space $\mathbb{C}P^1$. There also exist important projective multiplets that are defined only on an open domain of $\mathbb{C}P^1$. Before introducing them, let us give a few definitions. We define the north chart of $\mathbb{C}P^1$ to consist of those points for which the first component of $v^i = \left(v^{\underline{1}}, v^{\underline{2}}\right)$ is non-zero, $v^{\underline{1}} \neq 0$. The north chart of $\mathbb{C}P^1$ may be parametrized by the inhomogeneous complex coordinate $\zeta = v^{\underline{2}}/v^{\underline{1}} \in \mathbb{C}$. The only point of $\mathbb{C}P^1$ outside the north chart is characterized by $v^i_\infty = \left(0, v^{\underline{2}}\right)$ and describes an infinitely separated point. Thus we may think of the projective space $\mathbb{C}P^1$ as $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. The south chart of $\mathbb{C}P^1$ is defined to consist of those points for which the second component of $v^i = \left(v^{\underline{1}}, v^{\underline{2}}\right)$ is non-zero, $v^{\underline{2}} \neq 0$. The south chart is naturally parametrized by $1/\zeta$. The intersection of the north and south charts is $\mathbb{C} \setminus \{0\}$.

$\mathcal{O}(n)$ 多重态 (即 $H^{(n)}(v)$) 在整个投影空间 $\mathbb{C}P^1$ 上有良好定义。也存在一些重要的投影多重态，它们仅在 $\mathbb{C}P^1$ 的开区域上有定义。在引入这些多重态之前，我们先给出几个定义。我们将 $\mathbb{C}P^1$ 的北坐标卡定义为满足 $v^i = \left(v^{\underline{1}}, v^{\underline{2}}\right)$ 的第一分量非零的点的集合，即 $v^{\underline{1}} \neq 0$ 。$\mathbb{C}P^1$ 的北坐标卡可以用非齐次复坐标 $\zeta = v^{\underline{2}}/v^{\underline{1}} \in \mathbb{C}$ 参数化。$\mathbb{C}P^1$ 中唯一不在北坐标卡内的点由 $v^i_\infty = \left(0, v^{\underline{2}}\right)$ 刻画，对应一个无穷远点。因此我们可以将投影空间 $\mathbb{C}P^1$ 视为 $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ 。$\mathbb{C}P^1$ 的南坐标卡定义为满足 $v^i = \left(v^{\underline{1}}, v^{\underline{2}}\right)$ 的第二分量非零的点的集合，即 $v^{\underline{2}} \neq 0$ 。南坐标卡自然可以用 $1/\zeta$ 参数化。南北坐标卡的交集为 $\mathbb{C} \setminus \{0\}$ 。

An off-shell (charged) hypermultiplet can be described in terms of the so-called arctic weight- n multiplet $\Upsilon^{(n)}(v)$ which is defined to be holomorphic in the north chart of $\mathbb{C}P^1$:

离壳 (带荷) 超多重态可以用所谓的北极权 n 多重态 $\Upsilon^{(n)}(v)$ 描述，该多重态在 $\mathbb{C}P^1$ 的北坐标卡上全纯:

$$\Upsilon^{(n)}(v) = \left(v^{\underline{1}}\right)^n \Upsilon^{[n]}(\zeta), \ \Upsilon^{[n]}(\zeta) = \sum_{k=0}^{\infty} \Upsilon_k \zeta^k. \tag{47}$$

Its smile-conjugate antarctic multiplet $\breve{\Upsilon}^{(n)}(v)$ has the explicit form

它的微笑共轭南极多重态 $\breve{\Upsilon}^{(n)}(v)$ 具有如下显式形式

$$\breve{\Upsilon}^{(n)}(v) = \left(v^{\underline{2}}\right)^n \breve{\Upsilon}^{[n]}(\zeta) = \left(v^{\underline{1}}\zeta\right)^n \breve{\Upsilon}^{[n]}(\zeta), \ \breve{\Upsilon}^{[n]}(\zeta) = \sum_{k=0}^{\infty} \bar{\Upsilon}_k \frac{(-1)^k}{\zeta^k} \tag{48}$$

and is holomorphic in the south chart of $\mathbb{C}P^1$. The arctic multiplet can be coupled to a Yang-Mills multiplet in a complex representation of the gauge group [43]. The pair consisting of $\Upsilon^{[n]}(\zeta)$ and $\breve{\Upsilon}^{[n]}(\zeta)$ constitutes the so-called polar weight- n multiplet.

且在 $\mathbb{C}P^1$ 的南坐标卡上全纯。北极多重态可以耦合到对应规范群复表示的杨-米尔斯多重态 [43]。由 $\Upsilon^{[n]}(\zeta)$ 和 $\breve{\Upsilon}^{[n]}(\zeta)$ 构成的配对就是所谓的极权 n 多重态。

Our last example is a real tropical multiplet $\mathcal{U}^{(2n)}(v)$ of weight $2n$ defined by

我们的最后一个例子是权为 $2n$ 的实热带多重态 $\mathcal{U}^{(2n)}(v)$，其定义为

$$\mathcal{U}^{(2n)}(v) = \left(\mathrm{i}v^1 v^2\right)^n \mathcal{U}^{[2n]}(\zeta) = \left(v^1\right)^{2n} (\mathrm{i}\zeta)^n \mathcal{U}^{[2n]}(\zeta),$$

$$\mathcal{U}^{[2n]}(\zeta) = \sum_{k=-\infty}^{\infty} \mathcal{U}_k \zeta^k, \ \overline{\mathcal{U}}_k = (-1)^k \mathcal{U}_{-k} \tag{49}$$

This multiplet is holomorphic in the intersection of the north and south charts of the projective space $\mathbb{C}P^1$.

该多重态在投影空间 $\mathbb{C}P^1$ 的南北坐标卡的交集上全纯。

It should be pointed out that the modern projective-superspace terminology was introduced in [34].

需要指出，现代投影超空间的术语是在文献 [34] 中引入的。

Non-superconformal Projective Multiplets

非超共形投影多重态

In the original papers [42, 43] , general projective multiplets were introduced for the case of $\mathcal{N} = 2$ Poincaré supersymmetry, while definition (43) corresponds to superconformal projective multiplets. To define the former, the conformal Killing supervector field ξ in (43) should be replaced by a Killing supervector field

在原始论文 [42, 43] 中，广义投影多重态是针对 $\mathcal{N}=2$ 庞加莱超对称的情况引入的，而定义 (43) 对应超共形投影多重态。要定义前者，需要将 (43) 中的共形 Killing 超向量场 ξ 替换为 Killing 超向量场

$$\Xi=\Xi^{b}\partial_{b}+\Xi_{j}^{\beta}D_{\beta}^{j}+\overline{\Xi}_{\dot\beta}^{j}\bar{D}_{j}^{\dot\beta}=\overline{\Xi}. \tag{50}$$

By definition, Ξ is a conformal Killing supervector field such that the parameters (11b) and (11c) vanish. Its components are obtained from (14) by switching several parameters off:

根据定义，Ξ 是满足参数 (11b) 和 (11c) 为零的共形 Killing 超向量场，其分量可通过关闭若干参数从 (14) 得到:

$$\Xi_{+}^{\dot\alpha\alpha}=a^{\dot\alpha\alpha}+\bar{K}^{\dot\alpha}{}_{\dot\beta}y^{\dot\beta\alpha}+y^{\dot\alpha\beta}K_{\beta}{}^{\alpha}+4\mathrm{i}\bar{\varepsilon}^{\dot\alpha i}\theta_{i}^{\alpha}, \tag{51a}$$

$$\Xi_{i}^{\alpha}=\varepsilon_{i}^{\alpha}+\theta_{i}^{\beta}K_{\beta}^{\alpha}, \tag{51b}$$

where the complex four-vector Ξ_{+}^{a} is related to the vector component Ξ^{a} in (50) by the rule $\Xi_{+}^{a}=\Xi^{a}+2\mathrm{i}\Xi_{i}\sigma^{a}\overline{\theta}^{i}$. The super-Poincaré transformation law of a weight- n projective multiplet $Q^{(n)}(z,v)$ is obtained from (43c) by replacing $\xi\rightarrow\Xi$:

其中复四矢量 Ξ_{+}^{a} 与 (50) 中的矢量分量 Ξ^{a} 满足关系 $\Xi_{+}^{a}=\Xi^{a}+2\mathrm{i}\Xi_{i}\sigma^{a}\overline{\theta}^{i}$ 。权重为 n 的投影多重态 $Q^{(n)}(z,v)$ 的超庞加莱变换律可通过将 (43c) 中的 $\xi\rightarrow\Xi$ 替换得到:

$$\delta_{\Xi}Q^{(n)}=\Xi Q^{(n)}. \tag{52}$$

It is seen that the weight n of $Q^{(n)}$ becomes irrelevant from the point of view of the Poincaré supersymmetry. In particular, for the arctic (47) and antarctic (48) multiplets we can use the simplified notation

可见从庞加莱超对称的角度来看，$Q^{(n)}$ 的权重 n 不再相关。特别地，对于北极 (47) 和南极 (48) 多重态，我们可以使用简化记号

$$\Upsilon^{(n)}(v)=\left(v^{\underline{1}}\right)^{n}\Upsilon(\zeta),\ \breve{\Upsilon}^{(n)}(v)=\left(v^{\underline{1}}\zeta\right)^{n}\breve{\Upsilon}(\zeta). \tag{53}$$

Rigid Supersymmetric Sigma Models

刚性超对称西格玛模型

In order to get a better understanding of the opportunities provided by the projective multiplets, in this section we briefly discuss off-shell $\mathcal{N}=2$ supersymmetric sigma models in Minkowski superspace. We recall that the target spaces of $\mathcal{N}=2$ supersymmetric sigma models are hyperkähler manifolds in the super-Poincaré case [85] and hyperkähler cones in the superconformal case [86, 87] (see also [88] for the mathematical framework and [89] for a discussion in dimensions $3\leq d\leq 6$).

为了更好地理解投影多重态提供的研究可能，本节我们将简要讨论闵可夫斯基超空间中的离壳 $\mathcal{N}=2$ 超对称西格玛模型。我们已知，超庞加莱情形下，$\mathcal{N}=2$ 超对称西格玛模型的目标空间是超凯勒流形 [85]；超共形情形下，其目标空间是超凯勒锥 [86, 87](数学框架参见 [88]，维度 $3 \leq d \leq 6$ 下的相关讨论参见 [89])。

The $\mathcal{N} = 2$ supersymmetric action principle in projective superspace is formulated in terms of a Lagrangian $\mathcal{L}^{(2)}(z, v)$ which is a real weight-2 projective superfield. The action is

投影超空间中的 $\mathcal{N}=2$ 超对称作用量原理由拉格朗日量 $\mathcal{L}^{(2)}(z,v)$ 表述，该拉格朗日量是实权重为 2 的投影超场。作用量为

$$S := \frac{1}{2\pi}\oint_{\gamma}(v,\, \mathrm{d}v)\int \mathrm{d}^4x D^{(-4)}\mathcal{L}^{(2)}(z,v)\Big|_{\theta=\bar\theta=0},\ (v,\, \mathrm{d}v) := v^i\, \mathrm{d}v_i, \tag{54}$$

where γ denotes a closed integration contour and $D^{(-4)}$ is the fourth-order differential operator:

其中 γ 表示闭合积分围道，$D^{(-4)}$ 是四阶微分算子:

$$D^{(-4)} := \frac{1}{16}\left(D^{(-1)}\right)^2\left(\bar D^{(-1)}\right)^2,\ D^{(-1)}_{\alpha} := \frac{u_i D^i_\alpha}{(v,u)},\ \bar D^{(-1)}_{\dot\alpha} := \frac{u_i \bar D^i_{\dot\alpha}}{(v,u)}. \tag{55}$$

We recall that u_i is a fixed isotwistor chosen to be arbitrary modulo the condition $(v, u) \neq 0$ along the integration contour. The action is invariant under arbitrary projective transformations of the form:

我们已知 u_i 是固定同扭量，除了沿积分围道满足条件 $(v,u)\neq 0$ 外可以任意选取。作用量在下述任意投影变换下具有不变性:

$$\left(u^i, v^i\right) \to \left(u^i, v^i\right)\mathfrak{R},\ \mathfrak{R} = \begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{b} & \mathfrak{c} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}). \tag{56}$$

This gauge-like symmetry implies that the action is independent of $u_i, \delta_u S = 0$. It is also invariant under $\mathcal{N} = 2$ supersymmetry transformations

这类规范类对称性说明作用量不依赖于 $u_i, \delta_u S = 0$ ，同时它在 $\mathcal{N}=2$ 超对称变换下也不变

$$\delta_{\text{SUSY}}\mathcal{L}^{(2)} = \left(\varepsilon^\alpha_i Q^i_\alpha + \bar\varepsilon^i_{\dot\alpha}\bar Q^{\dot\alpha}_i\right)\mathcal{L}^{(2)}. \tag{57}$$

The projective-superspace action was originally given in [37] in a form that differs slightly from (54). The latter representation appeared first in [90].

投影超空间作用量最初出现在文献 [37] 中，形式与式 (54) 略有区别，后者的表示形式最早见于 [90]。

The action (54) is superconformal if the Lagrangian $\mathcal{L}^{(2)}$ is a superconformal weight-2 projective multiplet; see [55,84] for the proof. As an example, we consider an off-shell nonlinear σ -model described by n superconformal weight-1 arctic multiplets $\Upsilon^{(1)I}$ and their smile-conjugates $\breve\Upsilon^{(1)\bar I}$ with Lagrangian [84]

若拉格朗日量 $\mathcal{L}^{(2)}$ 是超共形权重为 2 的投影多重态，则作用量 (54) 是超共形的；证明参见 [55, 84]。我们举一个例子: 考虑由 n 个超共形权重为 1 的北极多重态 $\Upsilon^{(1)I}$ 及其微笑共轭 $\breve{\Upsilon}^{(1)\bar{I}}$ 描述的离壳非线性 σ 模型，其拉格朗日量为 [84]

$$\mathcal{L}^{(2)} = \mathrm{i}\mathcal{L}\left(\Upsilon^{(1)}, \breve{\Upsilon}^{(1)}\right), \tag{58a}$$

$$2\mathcal{L}\left(\Upsilon^{(1)}, \breve{\Upsilon}^{(1)}\right) = \left(\Upsilon^{(1)I}\frac{\partial}{\partial\Upsilon^{(1)I}} + \breve{\Upsilon}^{(1)\bar{I}}\frac{\partial}{\partial\breve{\Upsilon}^{(1)\bar{I}}}\right)\mathcal{L}\left(\Upsilon^{(1)}, \breve{\Upsilon}^{(1)}\right). \tag{58b}$$

In order for $\mathcal{L}^{(2)}$ to be real, it suffices to choose

要使 $\mathcal{L}^{(2)}$ 为实，只需选取

$$\mathcal{L}\left(\Upsilon^{(1)}, \breve{\Upsilon}^{(1)}\right) = \mathcal{K}\left(\Upsilon^{(1)}, \breve{\Upsilon}^{(1)}\right), \ \Phi^I\frac{\partial}{\partial\Phi^I}\mathcal{K}\left(\Phi, \overline{\Phi}\right) = \mathcal{K}\left(\Phi, \overline{\Phi}\right), \tag{59}$$

where $\mathcal{K}\left(\Phi, \overline{\Phi}\right)$ is a real analytic function of n complex variables Φ^I and their complex conjugates $\overline{\Phi}^{\bar{I}}$. The homogeneity properties of $\mathcal{K}$ imply that it can be interpreted as the Kähler potential of a Kähler cone [88].

其中 $\mathcal{K}\left(\Phi, \overline{\Phi}\right)$ 是 n 个复变量 Φ^I 及其复共轭 $\overline{\Phi}^{\bar{I}}$ 的实解析函数。$\mathcal{K}$ 的齐次性质表明，它可以被解释为凯勒锥的凯勒势 [88]。

The Lagrangian $\mathcal{L}^{(2)}$ in the general case of (58) is real if $\mathcal{L}\left(\Phi, \overline{\Omega}\right)$ obeys the reality condition $\overline{\mathcal{L}}\left(\overline{\Phi}, -\Omega\right) = -\mathcal{L}\left(\Phi, \overline{\Omega}\right)$, where $\overline{\mathcal{L}}\left(\overline{\Phi}, \Omega\right)$ denotes the complex conjugate of $\mathcal{L}\left(\Phi, \overline{\Omega}\right)$. A detailed study of the superconformal σ -model (59) was carried out in [84, 91] . That analysis was extended and generalized in [66] to the case of the most general superconformal σ -model (58).

在 (58) 的一般情形下，若 $\mathcal{L}\left(\Phi, \overline{\Omega}\right)$ 满足实条件 $\overline{\mathcal{L}}\left(\overline{\Phi}, -\Omega\right) = -\mathcal{L}\left(\Phi, \overline{\Omega}\right)$，则拉格朗日量 $\mathcal{L}^{(2)}$ 为实，其中 $\overline{\mathcal{L}}\left(\overline{\Phi}, \Omega\right)$ 是 $\mathcal{L}\left(\Phi, \overline{\Omega}\right)$ 的复共轭。超共形 σ 模型 (59) 的详细研究已经在 [84, 91] 中完成，该分析后来在 [66] 中被推广拓展到最一般的超共形 σ 模型 (58)。

Without loss of generality, we can assume that the integration contour γ does not pass through the "north pole" $v^i \sim (0,1)$. This chart is naturally parametrized by the inhomogeneous complex coordinate ζ defined by $v^i = v^{\underline{1}}(1,\zeta)$. Since the action is independent of u_i , the latter can be chosen to be $u_i = (1,0)$, such that $(v,u) = v^{\underline{1}} \neq 0$. We also represent the Lagrangian in the form:

不失一般性，我们可以假设积分围道 γ 不经过“北极点” $v^i \sim (0,1)$。该坐标卡自然由 $v^i = v^{\underline{1}}(1,\zeta)$ 定义的非齐次复坐标 ζ 参数化。由于作用量与 u_i 无关，可将后者取为 $u_i = (1,0)$，使得 $(v,u) = v^{\underline{1}} \neq 0$。我们也将拉格朗日写为如下形式:

$$\mathcal{L}^{(2)}(z,v) = \mathrm{i}v^{\underline{1}}v^{\underline{2}}\mathcal{L}(z,\zeta) = \mathrm{i}\left(v^{\underline{1}}\right)^2\zeta\mathcal{L}(z,\zeta), \ \breve{\mathcal{L}} = \mathcal{L}. \tag{60}$$

Now, the action takes the form:

现在，作用量形式为:

$$S = \frac{1}{16}\oint_{\gamma}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\int \mathrm{d}^4x\zeta\left(D^{\underline{1}}\right)^2\left(\bar{D}_{\underline{2}}\right)^2\mathcal{L}\left(z,\zeta\right)\Big|_{\theta_i=\bar{\theta}^i=0}. \tag{61}$$

Finally, the analyticity constraints (43a) on $\mathcal{L}\propto\mathcal{L}^{(2)}$ are equivalent to

最后，$\mathcal{L}\propto\mathcal{L}^{(2)}$ 上的解析性约束 (43a) 等价于

$$D^{\underline{2}}_{\alpha}\mathcal{L}\left(\zeta\right)=\zeta D^{\underline{1}}_{\alpha}\mathcal{L}\left(\zeta\right),\ \bar{D}^{\dot{\alpha}}_{\underline{2}}\mathcal{L}\left(\zeta\right)=-\frac{1}{\zeta}\bar{D}^{\dot{\alpha}}_{\underline{1}}\mathcal{L}\left(\zeta\right); \tag{62}$$

Hence, the action turns into

因此，作用量化为

$$S=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma}\frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^{4|4}z\mathcal{L}\left(z,\zeta\right)\Big|_{\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0},\ \mathrm{d}^{4|4}z\ :=\ \mathrm{d}^4x\ \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}. \tag{63}$$

Here the integration is carried out over the $\mathcal{N}=1$ Minkowski superspace with Grassmann coordinates $\theta^{\alpha}\equiv\theta^{\alpha}_{1}$ and $\bar{\theta}_{\dot{\alpha}}\equiv\bar{\theta}^{1}_{\dot{\alpha}}$. The action is now formulated entirely in terms of $\mathcal{N}=1$ superfields. By construction, it is off-shell $\mathcal{N}=2$ supersymmetric! This is one of the most powerful features of the projective-superspace approach. In what follows, we assume that $\theta^{\alpha}_{2}=0$ and $\bar{\theta}^{\dot{2}}_{\dot{\alpha}}=0$.

此处积分在 $\mathcal{N}=1$ 闵可夫斯基超空间上进行，该超空间具有格拉斯曼坐标 $\theta^{\alpha}\equiv\theta^{\alpha}_{1}$ 和 $\bar{\theta}_{\dot{\alpha}}\equiv\bar{\theta}^{1}_{\dot{\alpha}}$ 。现在作用量完全用 $\mathcal{N}=1$ 超场表述。根据构造，它是脱壳 $\mathcal{N}=2$ 超对称的！这是投影超空间方法最强大的特点之一。在下文中，我们假设 $\theta^{\alpha}_{2}=0$ 且 $\bar{\theta}^{\dot{2}}_{\dot{\alpha}}=0$ 。

The most general off-shell $\mathcal{N}=2$ supersymmetric nonlinear σ -model in projective superspace [42] can be realized in terms of polar supermultiplets

投影超空间中最一般的脱壳 $\mathcal{N}=2$ 超对称非线性 σ 模型 [42] 可以用极超多重态实现

$$S\left[\Upsilon,\breve{\Upsilon}\right]=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma}\frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^{4|4}z\mathcal{L}\left(\Upsilon^{I},\breve{\Upsilon}^{\bar{J}},\zeta\right), \tag{64}$$

where the arctic $\Upsilon\left(\zeta\right)$ and antarctic $\breve{\Upsilon}\left(\zeta\right)$ dynamical variables are generated by an infinite set of ordinary $\mathcal{N}=1$ superfields:

其中北极 $\Upsilon\left(\zeta\right)$ 和南极 $\breve{\Upsilon}\left(\zeta\right)$ 动力学变量由无穷多组普通 $\mathcal{N}=1$ 超场生成:

$$\Upsilon\left(\zeta\right)=\sum_{n=0}^{\infty}\Upsilon_n\zeta^n=\Phi+\sum\zeta+O\left(\zeta^2\right), \tag{65a}$$

$$\breve{\Upsilon}\left(\zeta\right)=\sum_{n=0}^{\infty}\bar{\Upsilon}_n(-\zeta)^{-n}=\bar{\Phi}-\frac{1}{\zeta}\bar{\sum}+O\left(\zeta^{-2}\right). \tag{65b}$$

As follows from the analyticity conditions (43a) (see also (62)), $\Phi := \Upsilon_0$ is chiral, $\bar{D}_{\dot{\alpha}}\Phi = 0, \sum := \Upsilon_1$ is complex linear, $\bar{D}^2 \sum = 0$, while the remaining components, $\Upsilon_2, \Upsilon_3, \ldots$, are unconstrained complex $\mathcal{N} = 1$ superfields. The latter superfields are auxiliary, since they appear in the action without derivatives.

由解析性条件 (43a)(也见 (62)) 可知，$\Phi := \Upsilon_0$ 是手征的，$\bar{D}_{\dot{\alpha}}\Phi = 0, \sum := \Upsilon_1$ 是复线性的，$\bar{D}^2 \sum = 0$ ，而其余分量 $\Upsilon_2, \Upsilon_3, \ldots$ 是无约束的复 $\mathcal{N} = 1$ 超场。这些超场是辅助场，因为它们出现在作用量中时不带有导数。

Although the σ -model (64) was first introduced in 1988 [42], for some 10 years it remained a purely formal construction, because there existed no technique to eliminate the auxiliary superfields contained in Υ^I , except in the case of Lagrangians quadratic in Υ^I and $\breve{\Upsilon}^{\bar{I}}$. This situation changed in the late 1990s when Refs. [46, 92, 93] identified a subclass of models (64) with interesting geometric properties. They correspond to the special case

尽管 σ 模型 (64) 最早于 1988 年提出 [42]，约 10 年间它都只是一个纯形式化构造，因为除了 Υ^I 和 $\breve{\Upsilon}^{\bar{I}}$ 二次型的拉格朗日情形外，没有技术可以消去 Υ^I 中包含的辅助超场。这种情况在 1990 年代末发生改变，文献 [46, 92, 93] 中识别出模型 (64) 的一个具有有趣几何性质的子类。它们对应如下特殊情形

$$\mathcal{L}\left(\Upsilon^I, \breve{\Upsilon}^{\bar{J}}, \zeta\right) = K\left(\Upsilon^I, \breve{\Upsilon}^{\bar{J}}\right), \tag{66}$$

where $K\left(\Phi^I, \overline{\Phi}^{\bar{J}}\right)$ is the Kähler potential of a Kähler manifold $\mathcal{M}$. The Kähler invariance $K\left(\Phi, \overline{\Phi}\right) \to K\left(\Phi, \overline{\Phi}\right) + \Lambda(\Phi) + \overline{\Lambda}\left(\overline{\Phi}\right)$ of the $\mathcal{N} = 1$ supersymmetric σ -model [94],

其中 $K\left(\Phi^I, \overline{\Phi}^{\bar{J}}\right)$ 是凯勒流形 $\mathcal{M}$ 的凯勒势。$\mathcal{N} = 1$ 超对称 σ 模型的凯勒不变性 $K\left(\Phi, \overline{\Phi}\right) \to K\left(\Phi, \overline{\Phi}\right) + \Lambda(\Phi) + \overline{\Lambda}\left(\overline{\Phi}\right)$ [94]

$$S\left[\Phi, \overline{\Phi}\right] = \int \mathrm{d}^{4|4}z K\left(\Phi^I, \overline{\Phi}^{\bar{J}}\right), \tag{67}$$

turns into

化为

$$K\left(\Upsilon, \breve{\Upsilon}\right) \to K\left(\Upsilon, \breve{\Upsilon}\right) + \Lambda(\Upsilon) + \overline{\Lambda}\left(\breve{\Upsilon}\right) \tag{68}$$

for the model

对该模型成立

$$S\left[\Upsilon, \breve{\Upsilon}\right] = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^{4|4}z K\left(\Upsilon^I, \breve{\Upsilon}^{\bar{J}}\right). \tag{69}$$

A holomorphic reparametrization of the Kähler manifold, $\Phi^I \to \Phi'^I = f^I(\Phi)$, has the following counterpart $\Upsilon^I(\zeta) \to \Upsilon'^I(\zeta) = f^I(\Upsilon(\zeta))$ in the $\mathcal{N} = 2$ case. Therefore, the physical superfields of the $\mathcal{N} = 2$

theory, Φ^I and $\sum^I$, should be regarded as coordinates of a point in the Kähler manifold and a tangent vector at the same point, respectively. Thus, the variables $\left(\Phi^I, \sum^J\right)$ parametrize the holomorphic tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$ [46].

凯勒流形的全纯重参数化 $\Phi^I \to \Phi'^I = f^I(\Phi)$ 在 $\mathcal{N} = 2$ 情形下有如下对应形式 $\Upsilon^I(\zeta) \to \Upsilon'^I(\zeta) = f^I(\Upsilon(\zeta))$ 。因此，$\mathcal{N} = 2$ 理论的物理超场 Φ^I 和 $\sum^I$ 应分别视为凯勒流形上一点的坐标，以及该点处的切向量。由此可知，变量 $\left(\Phi^I, \sum^J\right)$ 参数化了凯勒流形 $\mathcal{M}$ 的全纯切丛 $T\mathcal{M}$ [46]。

We assume that the auxiliary superfields in the model (64) have been eliminated. Then, the action (64) turns into

我们假设模型 (64) 中的辅助超场已被消去，此时作用量 (64) 变为

$$S = \int \mathrm{d}^{4|4}z \mathbb{L}\left(\Phi, \overline{\Phi}, \sum, \overline{\sum}\right), \tag{70}$$

for some Lagrangian $\mathbb{L}$. In the case of the model (69), $\mathbb{L}$ has the form [93]

对应某个拉格朗日量 $\mathbb{L}$ 。对于模型 (69)，$\mathbb{L}$ 的形式为 [93]

$$\mathbb{L}\left(\Phi, \overline{\Phi}, \sum, \overline{\sum}\right) = K\left(\Phi, \overline{\Phi}\right) + \sum_{n=1}^{\infty} \mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}\left(\Phi, \overline{\Phi}\right) \sum^{I_1} \dots \sum^{I_n} \overline{\sum}^{\bar{J}_1} \dots \overline{\sum}^{\bar{J}_n},$$

(71)

where $\mathcal{L}_{I\bar{J}} = -g_{I\bar{J}}\left(\Phi, \overline{\Phi}\right)$ and the coefficients $\mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}$, for $n > 1$, are tensor functions of the Kähler metric $g_{I\bar{J}}\left(\Phi, \overline{\Phi}\right) = \partial_I \partial_{\bar{J}} K\left(\Phi, \overline{\Phi}\right)$, the Riemann curvature $R_{I\bar{J}K\bar{L}}\left(\Phi, \overline{\Phi}\right)$, and its covariant derivatives. Each term in the action contains equal powers of $\sum$ and $\overline{\sum}$, since the original model (69) is invariant under rigid $\mathrm{U}(1)$ transformations [92]

其中 $\mathcal{L}_{I\bar{J}} = -g_{I\bar{J}}\left(\Phi, \overline{\Phi}\right)$,以及满足 $n > 1$ 的系数 $\mathcal{L}_{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}$,都是凯勒度量 $g_{I\bar{J}}\left(\Phi, \overline{\Phi}\right) = \partial_I \partial_{\bar{J}} K\left(\Phi, \overline{\Phi}\right)$ 、黎曼曲率 $R_{I\bar{J}K\bar{L}}\left(\Phi, \overline{\Phi}\right)$ 及其协变导数的张量函数。作用量中每一项所含 $\sum$ 和 $\overline{\sum}$ 的幂次相等，这是因为原模型 (69) 在刚性 $\mathrm{U}(1)$ 变换下不变 [92]

$$\Upsilon(\zeta) \mapsto \Upsilon\left(\mathrm{e}^{\mathrm{i}\alpha}\zeta\right) \Leftrightarrow \Upsilon_n(z) \mapsto \mathrm{e}^{\mathrm{i}n\alpha}\Upsilon_n(z). \tag{72}$$

To uncover the explicit structure of the hyperkähler target space associated with the σ -model (70), we should construct a dual formulation of the theory (70), obtained by dualizing each complex linear superfield $\sum^I$ and its conjugate $\overline{\sum}^{\bar{I}}$ into a chiral-antichiral pair Ψ_I and $\overline{\Psi}_{\bar{I}}$. In accordance with [42], this is accomplished through the use of the first-order action

为了揭示与 σ 模型 (70) 相关的超凯勒目标空间的显式结构，我们需要构造理论 (70) 的对偶表述: 将每个复线性超场 $\sum^I$ 及其共轭 $\overline{\sum}^{\bar{I}}$ 对偶化为手征-反手征对 Ψ_I 和 $\overline{\Psi}_{\bar{I}}$ 。根据文献 [42]，这一步可以通过引入一阶作用量实现

$$S_{\text{first-order}} = \int \mathrm{d}^{4|4}z \left\{ \mathbb{L}\left(\Phi, \overline{\Phi}, \sum, \overline{\sum}\right) + \Psi_I \sum{}^I + \overline{\Psi}_{\bar{I}} \overline{\sum}{}^{\bar{I}} \right\}. \tag{73}$$

Here $\sum^I$ is an unconstrained complex superfield, while Ψ_I is chiral, $\bar{D}_{\dot{\alpha}}\Psi_I = 0$. This model is equivalent to (70). Indeed, varying $S_{\text{first-order}}$ with respect to Ψ^I gives

此处 $\sum^I$ 是无约束复超场，Ψ_I 是手征超场，$\bar{D}_{\dot{\alpha}}\Psi_I = 0$ 。该模型等价于 (70): 对 Ψ^I 变分 $S_{\text{first-order}}$ 可得

$\bar{D}^2\sum^I = 0$ and then (73) reduces to the original theory, Eq. (70). On the other hand, we can integrate out $\sum$ s and their conjugates using their equations of motion

$\bar{D}^2\sum^I = 0$ ，此时 (73) 约化为原理论即式 (70)。另一方面，我们可以利用运动方程积分掉 $\sum$ s 及其共轭场

$$\frac{\partial}{\partial \sum^I} \mathbb{L}\left(\Phi, \overline{\Phi}, \sum, \overline{\sum}\right) + \Psi_I = 0, \tag{74}$$

which can be used to express the variables $\sum^I$ and their conjugates in terms of the other superfields, $\sum^I = \sum^I\left(\Phi, \Psi, \overline{\Phi}, \overline{\Psi}\right)$. This leads to the dual action

该运动方程可用来将变量 $\sum^I$ 及其共轭场用其他超场 $\sum^I = \sum^I\left(\Phi, \Psi, \overline{\Phi}, \overline{\Psi}\right)$ 表示，由此得到对偶作用量

$$S_{\text{dual}} = \int \mathrm{d}^{4|4}z \mathbb{K}\left(\Phi, \Psi, \overline{\Phi}, \overline{\Psi}\right). \tag{75}$$

Since this $\mathcal{N} = 2$ supersymmetric σ -model is formulated in terms of chiral $\mathcal{N} = 1$ superfields, its Lagrangian $\mathbb{K}\left(\Phi, \Psi, \overline{\Phi}, \overline{\Psi}\right)$ is the Kähler potential of a hyperkähler manifold [95] (or simply the hyperkähler potential).

由于该 $\mathcal{N} = 2$ 超对称 σ σ 模型是用手征 $\mathcal{N} = 1$ 超场表述的，其拉格朗日量 $\mathbb{K}\left(\Phi, \Psi, \overline{\Phi}, \overline{\Psi}\right)$ 是超凯勒流形的凯勒势 [95](也可直接称为超凯勒势)。

It may be shown [96] that the dual theory (75) is invariant under the second supersymmetry transformation

可以证明 [96]，对偶理论 (75) 在第二超对称变换下不变

$$\delta\Phi^I = \frac{1}{2}\bar{D}^2\left\{\bar{\varepsilon}\overline{\theta}\frac{\partial\mathbb{K}}{\partial\Psi_I}\right\}, \ \delta\Psi_I = -\frac{1}{2}\bar{D}^2\left\{\bar{\varepsilon}\overline{\theta}\frac{\partial\mathbb{K}}{\partial\Phi^I}\right\}. \tag{76}$$

These transformation laws follow from the linear supersymmetry (52) of the off-shell theory (64). If we introduce the condensed notation $\phi^a := \left(\Phi^I, \Psi_I\right)$ and $\overline{\phi}^{\bar{a}} = \left(\overline{\Phi}^{\bar{I}}, \overline{\Psi}_{\bar{I}}\right)$, as well as the symplectic matrices

这些变换律脱壳理论 (64) 的线性超对称变换 (52) 导出。若我们引入缩记符号 $\phi^a := \left(\Phi^I, \Psi_I\right)$ 和 $\overline{\phi}^{\bar{a}} = \left(\overline{\Phi}^{\bar{I}}, \overline{\Psi}_{\bar{I}}\right)$，以及辛矩阵

$$\mathbb{J}^{ab} = \mathbb{J}^{\bar{a}\bar{b}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{77}$$

then the supersymmetry transformation (76) can be rewritten as

则超对称变换 (76) 可改写为

$$\delta\phi^a = \frac{1}{2}\bar{D}^2\left\{\bar{\varepsilon}\bar{\theta}\mathbb{J}^{ab}\frac{\partial\mathbb{K}}{\partial\phi^b}\right\}, \tag{78}$$

which agrees with the general results of [95]. A remarkable result of Lindström and Roček [44] is the observation that the $\mathcal{N} = 2$ superfield Lagrangian in (64) can be identified with the generating function of a twisted symplectomorphism.

这与文献 [95] 的一般结果一致。Lindström 与 Roček 的一个著名结论 [44] 指出，(64) 中的 $\mathcal{N} = 2$ 超场拉格朗日量可以等同于扭辛微分同胚的生成函数。

In the case of the model (69), the hyperkähler potential has the form

对于模型 (69)，超凯勒势的形式为

$$\mathbb{K}\left(\Phi, \Psi, \overline{\Phi}, \overline{\Psi}\right) = K\left(\Phi, \overline{\Phi}\right) + \sum_{n=1}^{\infty} \mathcal{H}^{I_1\cdots I_n\bar{J}_1\cdots\bar{J}_n}\left(\Phi, \overline{\Phi}\right)\Psi_{I_1}\ldots\Psi_{I_n}\overline{\Psi}_{\bar{J}_1}\ldots\overline{\Psi}_{\bar{J}_n}$$

(79)

where $\mathcal{H}^{I\bar{J}}\left(\Phi, \overline{\Phi}\right) = g^{I\bar{J}}\left(\Phi, \overline{\Phi}\right)$. By construction, $\left(\Sigma^I, \overline{\Sigma}^{\bar{I}}\right)$ is a tangent vector at the point $\left(\Phi^I, \overline{\Phi}^{\bar{I}}\right)$ of $\mathcal{M}$; therefore, $\left(\Psi_I, \overline{\Psi}_{\bar{I}}\right)$ is a one-form at the same point. The variables $\phi^{\mathrm{a}} = \left(\Phi^I, \Psi_I\right)$ parametrize the holomorphic cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold $\mathcal{M}$ [92, 93] . The hyperkähler potential (79) was computed for all Hermitian symmetric spaces; see [97-99] and references therein.

其中 $\mathcal{H}^{I\bar{J}}\left(\Phi, \overline{\Phi}\right) = g^{I\bar{J}}\left(\Phi, \overline{\Phi}\right)$。根据构造，$\left(\Sigma^I, \overline{\Sigma}^{\bar{I}}\right)$ 是 $\mathcal{M}$ 上点 $\left(\Phi^I, \overline{\Phi}^{\bar{I}}\right)$ 处的切向量；因此 $\left(\Psi_I, \overline{\Psi}_{\bar{I}}\right)$ 是同一点处的 1-形式。变量 $\phi^{\mathrm{a}} = \left(\Phi^I, \Psi_I\right)$ 参数化了凯勒流形 $\mathcal{M}$ [92, 93] 的全纯余切丛 $T^*\mathcal{M}$ 。超凯勒势 (79) 已对所有埃尔米特对称空间算出；参见 [97-99] 及其中所列文献。

To conclude this section, we consider one more example of an off-shell σ -model, introduced in [100]. It is described by several real $\mathcal{O}(2)$ multiplets $H^{(2)I}(v)$, where $I = 1, \ldots, n$, which we represent as

作为本节小结，我们再讨论一个文献 [100] 中提出的脱壳 σ σ 模型例子。它由若干实 $\mathcal{O}(2)$ 多重态 $H^{(2)I}(v)$ 描述，其中 $I = 1, \ldots, n$ ，我们将其写为

$$H^{(2)I}(v) = \mathrm{i}\left(v^{\underline{1}}\right)^2 H^I(\zeta), \; H^I(\zeta) = \Phi^I + \zeta G^I - \zeta^2\overline{\Phi}^I. \tag{80}$$

The action functional is defined as follows:

作用量泛函定义如下:

$$S=-\frac{1}{2\pi\mathrm{i}}\oint_{\gamma}\frac{\mathrm{d}\zeta}{\zeta}\int\mathrm{d}^{4|4}z\frac{F\left(H^{I}\left(\zeta\right)\right)}{\zeta^{2}}+\text{ c.c,}\tag{81}$$

where γ is a closed contour around the origin and $F\left(z^{I}\right)$ is a holomorphic function of n variables. In accordance with the analyticity conditions (43), the $\mathcal{N}=1$ superfield Φ^{I} is chiral, $\bar{D}_{\dot{\alpha}}\Phi^{I}=0$, while the real superfield $G^{I}=\bar{G}^{I}$ is linear, $\bar{D}^{2}G^{I}=0$. The contour integral in (81) is easy to evaluate if we take into account that

其中 γ 是环绕原点的闭合围道，$F\left(z^{I}\right)$ 是 n 个变量的全纯函数。根据解析性条件 (43)，$\mathcal{N}=1$ 超场 Φ^{I} 是手征的，满足 $\bar{D}_{\dot{\alpha}}\Phi^{I}=0$ ，而实超场 $G^{I}=\bar{G}^{I}$ 是线性的，满足 $\bar{D}^{2}G^{I}=0$ 。只要注意到下述关系，(81) 中的围道积分很容易计算:

$$F\left(H\left(\zeta\right)\right)=F\left(\Phi\right)+\zeta F_{I}\left(\Phi\right)G^{I}-\zeta^{2}\left(F_{I}\left(\Phi\right)\overline{\Phi}^{I}-\frac{1}{2}F_{IJ}G^{I}G^{J}\right)+O\left(\zeta^{3}\right).\tag{82}$$

Only the ζ^{2} term in this expression contributes to the contour integral. Thus, we get

该表达式中只有 ζ^{2} 项对围道积分有贡献。因此我们得到

$$S\left[\Phi,\overline{\Phi},G\right]=\int\mathrm{d}^{4|4}z\left\{K\left(\Phi,\overline{\Phi}\right)-\frac{1}{2}G_{IJ}\left(\Phi,\overline{\Phi}\right)G^{I}G^{J}\right\},\tag{83a}$$

where we have defined

其中我们定义了

$$K\left(\Phi,\overline{\Phi}\right)=\overline{\Phi}^{I}F_{I}\left(\Phi\right)+\Phi^{I}\bar{F}_{I}\left(\overline{\Phi}\right),\ g_{IJ}\left(\Phi,\overline{\Phi}\right)=F_{IJ}\left(\Phi\right)+\bar{F}_{IJ}\left(\overline{\Phi}\right).\tag{83b}$$

We can interpret $K\left(\Phi,\overline{\Phi}\right)$ and $g_{IJ}\left(\Phi,\overline{\Phi}\right)$ as the Kähler potential of a Kähler manifold and the corresponding Kähler metric. Each linear superfield G^{I} in (83a) may be dualized into a chiral superfield Ψ_{I} and its conjugate $\overline{\Psi}_{I}$. As a result, the action turns into

我们可将 $K\left(\Phi,\overline{\Phi}\right)$ 和 $g_{IJ}\left(\Phi,\overline{\Phi}\right)$ 分别解释为凯勒流形的凯勒势与对应凯勒度量。式 (83a) 中的每个线性超场 G^{I} 都可以对偶化为手征超场 Ψ_{I} 及其共轭 $\overline{\Psi}_{I}$ ，最终作用量变为

$$S\left[\Phi,\Psi,\overline{\Phi},\overline{\Psi}\right]=\int\mathrm{d}^{4|4}z\mathbb{K}\left[\Phi,\Psi,\overline{\Phi},\overline{\Psi}\right],\tag{84a}$$

$$\mathbb{K}\left[\Phi,\Psi,\overline{\Phi},\overline{\Psi}\right]=K\left(\Phi,\overline{\Phi}\right)+\frac{1}{2}g^{IJ}\left(\Phi,\overline{\Phi}\right)\left(\Psi_{I}+\overline{\Psi}_{I}\right)\left(\Psi_{J}+\overline{\Psi}_{J}\right).\tag{84b}$$

Since the original action (81) is $\mathcal{N}=2$ supersymmetric, its dual (84a) is also $\mathcal{N}=2$ supersymmetric. Since the latter is formulated in terms of $\mathcal{N}=1$ super-fields, (84b) is the Kähler potential of a hyperkähler manifold. The correspondence $K\left(\Phi,\overline{\Phi}\right)\to\mathbb{K}\left[\Phi,\Psi,\overline{\Phi},\overline{\Psi}\right]$ constitutes the so-called rigid c-map [101,102].

由于原作用量 (81) 是 $\mathcal{N}=2$ 超对称的，其对偶形式 (84a) 也具有 $\mathcal{N}=2$ 超对称性。后者以 $\mathcal{N}=1$ 超场表述，因此 (84b) 是超凯勒流形的凯勒势。该对应关系 $K\left(\Phi,\overline{\Phi}\right)\to\mathbb{K}\left[\Phi,\Psi,\overline{\Phi},\overline{\Psi}\right]$ 构成了所谓的刚性 c 映射 [101,102]。

Conformal Superspace

共形超空间

In section "Rigid Superconformal Transformations" we have reviewed a simple approach to obtain the $\mathcal{N}=2$ superconformal algebra by employing the conformal Killing supervector fields of flat superspace. The goal of this section is to construct the gauge theory of the latter, known in the literature as conformal superspace. It was introduced in [63] as a generalization of the $\mathcal{N}=1$ case analyzed in [68]. This approach, which will be reviewed in the present section, is of particular importance as it provides powerful tools to construct manifestly gauge-invariant supergravity actions and to engineer general couplings of supergravity to matter.

我们在“刚性超共形变换”一节中回顾了一种利用平坦超空间的共形基灵超向量场得到 $\mathcal{N}=2$ 超共形代数的简单方法。本节的目标是构造后者的规范理论，也就是文献中所称的共形超空间。它在文献 [63] 中被提出，是对文献 [68] 中分析的 $\mathcal{N}=1$ 情况的推广。本节将回顾这种方法，它尤为重要，因为它提供了强大的工具来构造明显规范不变的超引力作用量，并设计超引力与物质的一般耦合。

Gauging the Superconformal Algebra in Superspace

超空间中超共形代数的规范化

Conformal superspace is a gauge theory of the superconformal algebra. It can be identified with a pair $\left(\mathcal{M}^{4|8},\nabla\right)$. Here $\mathcal{M}^{4|8}$ denotes a supermanifold parametrized by local coordinates $z^M=\left(x^m,\theta^\mu_l,\bar{\theta}^l_{\dot\mu}\right)$, and ∇ is a covariant derivative associated with the superconformal algebra. We recall that the generators $X_{\tilde{a}}$ of the supercon-formal algebra are given by eq. (17). They can be grouped in two disjoint subsets,

共形超空间是超共形代数的规范理论，它可以对应为一个对 $\left(\mathcal{M}^{4|8},\nabla\right)$。其中 $\mathcal{M}^{4|8}$ 表示由局部坐标 $z^M=\left(x^m,\theta^\mu_l,\bar{\theta}^l_{\dot\mu}\right)$ 参数化的超流形，∇ 是与超共形代数关联的协变导数。回顾可知，超共形代数的生成元 $X_{\tilde{a}}$ 由式 (17) 给出，它们可以分为两个不相交子集，

$$X_{\tilde{a}}=\left(P_A,X_{\underline{a}}\right),\ X_{\underline{a}}=\left(M_{ab},\mathbb{Y},J_{ij},\mathbb{D},K^A\right),\tag{85}$$

each of which constitutes a superalgebra:

每个子集各自构成一个超代数:

$$[P_A, P_B\} = -f_{AB}{}^C P_C \tag{86a}$$

$$\left[X_{\underline{a}}, X_{\underline{b}}\right\} = -f_{\underline{ab}}{}^{\underline{c}} X_{\underline{c}} \tag{86b}$$

$$\left[X_{\underline{a}}, P_B\right\} = -f_{\underline{a}B}{}^{\underline{c}} X_{\underline{c}} - f_{\underline{a}B}{}^C P_C. \tag{86c}$$

Here the structure constants $f_{AB}{}^C$ contain only one non-zero component, which is $f_{\alpha j}^{i}{}^{\dot\beta c} = 2\mathrm{i}\delta_j^i(\sigma^c)_\alpha{}^{\dot\beta}$.

此处结构常数 $f_{AB}{}^C$ 仅包含一个非零分量，即 $f_{\alpha j}^{i}{}^{\dot\beta c} = 2\mathrm{i}\delta_j^i(\sigma^c)_\alpha{}^{\dot\beta}$.

In order to define the covariant derivatives, $\nabla_A = \left(\nabla_a, \nabla_\alpha^i, \overline{\nabla}_i^{\dot\alpha}\right)$, we associate with each generator $X_{\underline{a}} = \left(M_{ab}, \mathbb{Y}, J_{ij}, \mathbb{D}, K^A\right) = \left(M_{ab}, \mathbb{Y}, J_{ij}, \mathbb{D}, K^a, S_i^\alpha, \bar S_{\dot\alpha}^i\right)$ a connection one-form $\omega^{\underline{a}} = \left(\Omega^{ab}, \Phi, \Theta^{ij}, B, \mathfrak{F}_A\right) = \left(\Omega^{ab}, \Phi, \Theta^{ij}, B, \mathfrak{F}_a, \mathfrak{F}_\alpha^i, \overline{\mathfrak{F}}_i^{\dot\alpha}\right) = \mathrm{d}z^M \omega_M{}^{\underline{a}}$, and with P_A a supervielbein one-form $E^A = \left(E^a, E_i^\alpha, \bar E_{\dot\alpha}^i\right) = \mathrm{d}z^M E_M{}^A$ (the latter will be often referred to as the vielbein). It is assumed that the supermatrix $E_M{}^A$ is nonsingular, $E := \mathrm{Ber}\left(E_M{}^A\right) \equiv \mathrm{sdet}\left(E_M{}^A\right) \neq 0$, and hence there exists a unique inverse supervielbein. The latter is given by the supervector fields $E_A = E_A{}^M(z)\partial_M$, with $\partial_M = \partial/\partial z^M$, which constitute a new basis for the tangent space at each point $z^M \in \mathcal{M}^{4|8}$. The supermatrices $E_A{}^M$ and $E_M{}^A$ satisfy the properties $E_A{}^M E_M{}^B = \delta_A{}^B$ and $E_M{}^A E_A{}^N = \delta_M{}^N$. With respect to the basis E^A, the connection is expressed as $\omega^{\underline{a}} = E^B \omega_B{}^{\underline{a}}$, where $\omega_B{}^{\underline{a}} = E_B{}^M \omega_M{}^{\underline{a}}$. The covariant derivative is given by

为定义协变导数 $\nabla_A = \left(\nabla_a, \nabla_\alpha^i, \overline{\nabla}_i^{\dot\alpha}\right)$，我们为每个生成元 $X_{\underline{a}} = \left(M_{ab}, \mathbb{Y}, J_{ij}, \mathbb{D}, K^A\right) = \left(M_{ab}, \mathbb{Y}, J_{ij}, \mathbb{D}, K^a, S_i^\alpha, \bar S_{\dot\alpha}^i\right)$ 关联一个联络一元型 $\omega^{\underline{a}} = \left(\Omega^{ab}, \Phi, \Theta^{ij}, B, \mathfrak{F}_A\right) = \left(\Omega^{ab}, \Phi, \Theta^{ij}, B, \mathfrak{F}_a, \mathfrak{F}_\alpha^i, \overline{\mathfrak{F}}_i^{\dot\alpha}\right) = \mathrm{d}z^M \omega_M{}^{\underline{a}}$，为 P_A 关联一个超 Vielbein 一元型 $E^A = \left(E^a, E_i^\alpha, \bar E_{\dot\alpha}^i\right) = \mathrm{d}z^M E_M{}^A$ (后者常简称为 Vielbein)。假定超矩阵 $E_M{}^A$ 非退化，满足 $E := \mathrm{Ber}\left(E_M{}^A\right) \equiv \mathrm{sdet}\left(E_M{}^A\right) \neq 0$，因此存在唯一的逆超 Vielbein。逆超 Vielbein 由超矢量场 $E_A = E_A{}^M(z)\partial_M$ 给出，满足 $\partial_M = \partial/\partial z^M$，它们构成每一点 $z^M \in \mathcal{M}^{4|8}$ 处切空间的一组新基。超矩阵 $E_A{}^M$ 和 $E_M{}^A$ 满足性质 $E_A{}^M E_M{}^B = \delta_A{}^B$ 和 $E_M{}^A E_A{}^N = \delta_M{}^N$。在基 E^A 下，联络可表示为 $\omega^{\underline{a}} = E^B \omega_B{}^{\underline{a}}$，其中 $\omega_B{}^{\underline{a}} = E_B{}^M \omega_M{}^{\underline{a}}$。协变导数由下式给出

$$\nabla_A = E_A - \omega_A{}^{\underline{b}} X_{\underline{b}} = E_A - \frac{1}{2}\Omega_A{}^{bc} M_{bc} - \mathrm{i}\Phi_A \mathbb{Y} - \Theta_A{}^{jk} J_{jk} - B_A \mathbb{D} - \mathfrak{F}_{AB} K^B.$$

(87)

It can be recast as a super one-form

它可以改写为一个超一元型

$$\nabla = \mathrm{d} - \omega^{\underline{a}} X_{\underline{a}} \ \nabla = E^A \nabla_A. \tag{88}$$

The translation generators P_B do not show up in (87) and (88). It is assumed that the operators ∇_A replace P_A and obey the graded commutation relations

平移生成元 P_B 未出现在 (87) 和 (88) 中。假定算符 ∇_A 替代 P_A，并满足分次对易关系

$$\left[X_{\underline{b}}, \nabla_A\right\} = -f_{\underline{b}A}{}^C \nabla_C - f_{\underline{b}A}{}^{\underline{c}} X_{\underline{c}} \tag{89}$$

compared with (86c). In particular, the algebra of K^A with ∇_B is given by

与 (86c) 相比，特别地，K^A 与 ∇_B 的代数由下式给出

$$[K^a, \nabla_b] = 2\delta^a_b \mathbb{D} + 2M^a{}_b, \tag{90a}$$

$$\left\{S^\alpha_i, \nabla^j_\beta\right\} = \delta^j_i \delta^\alpha_\beta (2\mathbb{D} - \mathbb{Y}) - 4\delta^j_i M^\alpha{}_\beta + 4\delta^\alpha_\beta J_i{}^j, \tag{90b}$$

$$\left\{\bar{S}^i_{\dot\alpha}, \overline{\nabla}^{\dot\beta}_j\right\} = \delta^i_j \delta^{\dot\beta}_{\dot\alpha} (2\mathbb{D} + \mathbb{Y}) + 4\delta^i_j \bar{M}_{\dot\alpha}{}^{\dot\beta} - 4\delta^{\dot\beta}_{\dot\alpha} J^i{}_j, \tag{90c}$$

$$\left[K^a, \nabla^i_\beta\right] = -\mathrm{i}(\sigma^a)_\beta{}^{\dot\beta} \bar{S}^i_{\dot\beta} \ \left[K^a, \overline{\nabla}^{\dot\beta}_i\right] = -\mathrm{i}(\sigma^a)^{\dot\beta}{}_\beta S^\beta_i, \tag{90d}$$

$$\left[S^\alpha_i, \nabla_b\right] = \mathrm{i}(\sigma_b)^\alpha{}_{\dot\beta} \overline{\nabla}^{\dot\beta}_i \ \left[\bar{S}^i_{\dot\alpha}, \nabla_b\right] = \mathrm{i}(\sigma_b)_{\dot\alpha}{}^\beta \nabla^i_\beta \tag{90e}$$

where all other graded commutators vanish.

其中所有其他分次对易子均为零

By definition, the gauge group of conformal supergravity is generated by local transformations of the form

根据定义，共形超引力的规范群由如下形式的局域变换生成

$$\delta_{\mathcal{K}} \nabla_A = [\mathcal{K}, \nabla_A] \tag{91a}$$

$$\mathcal{K} = \xi^B \nabla_B + \Lambda^{\underline{b}} X_{\underline{b}}$$

$$= \xi^B \nabla_B + \frac{1}{2} K^{bc} M_{bc} + \sum \mathbb{D} + \mathrm{i}\rho Y + \Lambda^{jk} J_{jk} + \Lambda_B K^B \tag{91b}$$

where the gauge parameters satisfy natural reality conditions. In applying Eq. (91), we interpret that

其中规范参数满足自然实条件。应用式 (91) 时我们约定

$$\nabla_A \xi^B \ := \ E_A \xi^B + \omega_A{}^{\underline{c}} \xi^D f_{D\underline{c}}{}^B \tag{92a}$$

$$\nabla_A \Lambda^{\underline{b}} \ := \ E_A \Lambda^{\underline{b}} + \omega_A{}^{\underline{c}} \xi^D f_{D\underline{c}}{}^{\underline{b}} + \omega_A{}^{\underline{c}} \Lambda^{\underline{d}} f_{\underline{dc}}{}^{\underline{b}}. \tag{92b}$$

Then it follows from (91) that

那么由 (91) 可推得

$$\delta_{\mathcal{K}} E^A = \mathrm{d}\xi^A + E^B \Lambda^{\underline{c}} f_{\underline{c}B}{}^A + \omega^{\underline{b}} \xi^C f_{C\underline{b}}{}^A + E^B \xi^C \mathcal{T}_{CB}{}^A, \tag{93a}$$

$$\delta_{\mathcal{K}} \omega^{\underline{a}} = \mathrm{d}\Lambda^{\underline{a}} + \omega^{\underline{b}} \Lambda^{\underline{c}} f_{\underline{cb}}{}^{\underline{a}} + \omega^{\underline{b}} \xi^C f_{C\underline{b}}{}^{\underline{a}} + E^B \Lambda^{\underline{c}} f_{\underline{c}B}{}^{\underline{a}} + E^B \xi^C \mathcal{R}_{CB}{}^{\underline{a}}.$$

(93b)

Here we have made use of the graded commutation relations

此处我们用到了分次对易关系

$$[\nabla_A, \nabla_B\} = -\mathcal{T}_{AB}{}^C \nabla_C - \mathcal{R}_{AB}{}^{\underline{c}} X_{\underline{c}} \tag{94}$$

where $\mathcal{T}_{AB}{}^C$ and $\mathcal{R}_{AB}{}^C$ denote the torsion and the curvature, respectively. They can be recast in terms of two-forms

其中 $\mathcal{T}_{AB}{}^C$ 和 $\mathcal{R}_{AB}{}^C$ 分别表示挠率和曲率，它们可以用二形式重新表示

$$\mathcal{T}^A := \frac{1}{2} E^C \wedge E^B \mathcal{T}_{BC}{}^A = \mathrm{d}E^A - E^C \wedge \omega^{\underline{b}} f_{\underline{b}C}{}^A \tag{95a}$$

$$\mathcal{R}^{\underline{a}} := \frac{1}{2} E^C \wedge E^B \mathcal{R}_{BC}{}^{\underline{a}} = \mathrm{d}\omega^{\underline{a}} - E^C \wedge \omega^{\underline{b}} f_{\underline{b}C}{}^{\underline{a}} - \frac{1}{2} \omega^{\underline{c}} \wedge \omega^{\underline{b}} f_{\underline{bc}}{}^{\underline{a}}. \tag{95b}$$

Making use of the graded Jacobi identity

利用分次雅可比恒等式

$$0 = (-1)^{\varepsilon_{\underline{a}} \varepsilon_C} \left[X_{\underline{a}}, [\nabla_B, \nabla_C\} \right\} + \text{ (two cycles)} \tag{96}$$

we derive the action of $X_{\underline{a}}$ on the geometric objects

我们推导出 $X_{\underline{a}}$ 在几何对象上的作用

$$X_{\underline{a}} \mathcal{T}_{BC}{}^D = -(-1)^{\varepsilon_{\underline{a}}(\varepsilon_B + \varepsilon_C)} \mathcal{T}_{BC}{}^E f_{E\underline{a}}{}^D - 2 f_{\underline{a}[B}{}^E \mathcal{T}_{|E|C\}}{}^D - 2 f_{\underline{a}[B}{}^{\underline{e}} f_{|\underline{e}|C]}{}^D$$

(97a)

$$X_{\underline{a}} \mathcal{R}_{BC}{}^{\underline{d}} = -(-1)^{\varepsilon_{\underline{a}}(\varepsilon_B + \varepsilon_C)} \left(\mathcal{T}_{BC}{}^E f_{E\underline{a}}{}^{\underline{d}} + \mathcal{R}_{BC}{}^{\underline{e}} f_{\underline{ea}}{}^{\underline{d}} \right) - 2 f_{\underline{a}[B}{}^E \mathcal{R}_{|E|C\}}{}^{\underline{d}}$$

$$-2 f_{\underline{a}[B}{}^{\underline{e}} f_{|\underline{e}|C]}{}^{\underline{d}}. \tag{97b}$$

The supergravity gauge group acts on a conformal tensor superfield U (with indices suppressed) as

超引力规范群按如下方式作用在共形张量超场 U (指标省略) 上

$$\delta_{\mathcal{K}} U = \mathcal{K} U \tag{98}$$

The torsion $\mathcal{T}_{AB}{}^{C}$ and the curvature $\mathcal{R}_{AB}{}^{\underline{c}}$ are conformal tensor superfields. Of special significance are primary superfields. A tensor superfield U (with suppressed indices) is said to be primary if it is characterized by the properties

挠率 $\mathcal{T}_{AB}{}^{C}$ 和曲率 $\mathcal{R}_{AB}{}^{\underline{c}}$ 都是共形张量超场。其中本原超场具有特殊意义。若张量超场 U (指标省略) 满足下述性质，则称它为本原超场

$$K^{A}U = 0, \ \mathbb{D}U = wU, \ \mathbb{Y}U = cU, \tag{99}$$

for some real constants w and c , which are called the dimension (or Weyl weight) and $\mathrm{U}(1)_R$ charge of U , respectively. From algebra (26b), it is seen that if a superfield is annihilated by the S -supersymmetry generators, then it is necessarily primary.

对应某些实常数 w 和 c ，二者分别称为 U 的维数 (或外尔权) 和 $\mathrm{U}(1)_R$ 荷。由 (26b) 的代数关系可知，如果一个超场被 S 超对称生成元零化，则它必然是本原超场

Let us summarize some important features of the gauging procedure. In curved superspace, the superconformal algebra (86) is replaced with

我们总结一下规范过程的若干重要性质。在弯曲超空间中，超共形代数 (86) 被替换为

$$\left[X_{\underline{a}}, X_{\underline{b}}\right\} = -f_{\underline{a}\underline{b}}{}^{\underline{c}}X_{\underline{c}} \tag{100a}$$

$$\left[X_{\underline{a}}, \nabla_B\right\} = -f_{\underline{a}B}{}^{C}\nabla_C - f_{\underline{a}B}{}^{\underline{c}}X_{\underline{c}} \tag{100b}$$

$$[\nabla_A, \nabla_B\} = -\mathcal{T}_{AB}{}^{C}\nabla_C - \mathcal{R}_{AB}{}^{\underline{c}}X_{\underline{c}}. \tag{100c}$$

Here the torsion and curvature tensors obey Bianchi identities which follow from

此处挠率张量和曲率张量满足比安基恒等式，该恒等式由下述关系导出

$$0 = (-1)^{\varepsilon_A\varepsilon_C}\,[\nabla_A, [\nabla_B, \nabla_C\}\} + \text{ (two cycles)}\,. \tag{101}$$

Unlike (86), which is determined by the structure constants, the graded commutation relations (100) involve structure functions $\mathcal{T}_{AB}{}^{C}$ and $\mathcal{R}_{AB}{}^{c}$.

与由结构常数确定的 (86) 不同，分次对易关系 (100) 包含结构函数 $\mathcal{T}_{AB}{}^{C}$ 和 $\mathcal{R}_{AB}{}^{c}$

Conventional Constraints for Weyl Multiplet

魏尔多重态的常规约束

The framework described in the previous subsection defines a geometric setup to obtain a multiplet of conformal supergravity containing the metric. However, in general, the resulting multiplet is reducible. To obtain an irreducible multiplet, it is necessary to impose constraints on the torsion and curvatures appearing in Eq. (94). This is a standard task in geometric superspace approaches to supergravity, and it is pedagogically reviewed in [76, 103]. One beautiful feature of the construction of [63] is the simplicity of the superspace constraints needed to obtain the Weyl multiplet of conformal supergravity. In fact, to obtain a sufficient set of constraints, one requires the algebra (94) to have a Yang-Mills structure. Specifically, one imposes

上一小节描述的框架给出了一个几何构造，用于得到包含度规的共形超引力多重态。但一般来说，这样得到的多重态是可约的。要得到不可约多重态，必须对式 (94) 中出现的挠率和曲率施加约束。这是超引力几何超空间方法中的标准任务，在文献 [76, 103] 中有教学性综述。文献 [63] 构造的一个优美特点是，得到共形超引力魏尔多重态所需的超空间约束非常简单。事实上，要得到一组充分的约束，只需要求代数 (94) 具有杨-米尔斯结构即可。具体来说，我们要求

$$\left\{\nabla_\alpha^i,\nabla_\beta^j\right\}=-2\varepsilon^{ij}\varepsilon_{\alpha\beta}\overline{\mathcal{W}},\ \left\{\overline{\nabla}_i^{\dot\alpha},\overline{\nabla}_j^{\dot\beta}\right\}=2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\mathcal{W}, \tag{102a}$$

$$\left\{\nabla_\alpha^i,\overline{\nabla}_j^{\dot\beta}\right\}=-2\mathrm{i}\delta_j^i\nabla_\alpha{}^{\dot\beta}, \tag{102b}$$

where the operator $\overline{\mathcal{W}}$ is the complex conjugate of $\mathcal{W}$. The latter takes the form

其中算符 $\overline{\mathcal{W}}$ 是 $\mathcal{W}$ 的复共轭。后者的形式为

$$\mathcal{W}=\frac{1}{2}\mathcal{W}(M)^{ab}M_{ab}+\mathrm{i}\mathcal{W}(\mathbb{Y})\mathbb{Y}+\mathcal{W}(J)^{ij}J_{ij}+\mathcal{W}(\mathbb{D})\mathbb{D}$$

$$+\mathcal{W}(S)_\alpha^iS_i^\alpha+\mathcal{W}(\bar S)_i^{\dot\alpha}\bar S_{\dot\alpha}^i+\mathcal{W}(K)_aK^a. \tag{103}$$

Having imposed the constraints (102a), the Bianchi identities (101) become nontrivial and now play the role of consistency conditions which may be used to determine the torsion and curvature. Their solution, up to mass dimension-3/2, is as follows:

施加约束 (102a) 后，比安基恒等式 (101) 变得非平庸，现在它起到一致性条件的作用，可以用来确定挠率和曲率。直到质量维数 3/2，其解如下:

$$\left\{\nabla_\alpha^i,\nabla_\beta^j\right\}=2\varepsilon^{ij}\varepsilon_{\alpha\beta}\left(\bar W_{\dot\gamma\dot\delta}\bar M^{\dot\gamma\dot\delta}+\frac{1}{4}\overline{\nabla}_{\dot\gamma k}\bar W^{\dot\gamma\dot\delta}\bar S_{\dot\delta}^k-\frac{1}{4}\nabla_{\gamma\dot\delta}\bar W^{\dot\delta}{}_{\dot\gamma}K^{\gamma\dot\gamma}\right), \tag{104a}$$

$$\left\{\overline{\nabla}_i^{\dot\alpha},\overline{\nabla}_j^{\dot\beta}\right\}=-2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\left(W^{\gamma\delta}M_{\gamma\delta}-\frac{1}{4}\nabla^{\gamma k}W_{\gamma\delta}S_k^\delta+\frac{1}{4}\nabla^{\gamma\dot\gamma}W_\gamma^\delta K_{\delta\dot\gamma}\right), \tag{104b}$$

$$\left\{\nabla_\alpha^i,\overline{\nabla}_j^{\dot\beta}\right\}=-2\mathrm{i}\delta_j^i\nabla_\alpha{}^{\dot\beta}, \tag{104c}$$

$$\left[\nabla_{\alpha\dot\alpha},\nabla_\beta^i\right]=-\mathrm{i}\varepsilon_{\alpha\beta}\bar W_{\dot\alpha\dot\beta}\overline{\nabla}^{\dot\beta i}-\frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\overline{\nabla}^{\dot\beta i}\bar W_{\dot\alpha\dot\beta}\mathbb{D}-\frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\overline{\nabla}^{\dot\beta i}\bar W_{\dot\alpha\dot\beta}\mathbb{Y}+\mathrm{i}\varepsilon_{\alpha\beta}\overline{\nabla}_j^{\dot\beta}\bar W_{\dot\alpha\dot\beta}J^{ij}$$

$$-\mathrm{i}\varepsilon_{\alpha\beta}\overline{\nabla}_{\dot\beta}^i\bar W_{\dot\gamma\dot\alpha}\bar M^{\dot\beta\dot\gamma}-\frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\overline{\nabla}_{\dot\alpha}^i\overline{\nabla}_k^{\dot\beta}\bar W_{\dot\beta\dot\gamma}\bar S^{\dot\gamma k}+\frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\gamma\dot\beta}\bar W_{\dot\alpha\dot\beta}S_\gamma^i$$

$$+\frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar{\nabla}^i_{\dot{\alpha}}\nabla^{\gamma}{}_{\dot{\gamma}}\bar{W}^{\dot{\gamma}\dot{\beta}}K_{\gamma\dot{\beta}} \tag{104d}$$

$$\left[\nabla_{\alpha\dot{\alpha}},\bar{\nabla}^{\dot{\beta}}_i\right]=\mathrm{i}\delta^{\dot{\beta}}_{\dot{\alpha}}W_{\alpha\beta}\nabla^{\beta}_i+\frac{\mathrm{i}}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla^{\beta}_i W_{\alpha\beta}\mathbb{D}-\frac{\mathrm{i}}{4}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla^{\beta}_i W_{\alpha\beta}\mathbb{Y}+\mathrm{i}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla^{\beta j}W_{\alpha\beta}J_{ij}$$

$$+\mathrm{i}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla^{\beta}_i W^{\gamma}{}_{\alpha}M_{\beta\gamma}+\frac{\mathrm{i}}{4}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla_{\alpha i}\nabla^{\beta j}W_{\beta}{}^{\gamma}S_{\gamma j}-\frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla^{\beta}{}_{\dot{\gamma}}W_{\alpha\beta}\bar{S}^{\dot{\gamma}}_i$$

$$+\frac{\mathrm{i}}{4}\delta^{\dot{\beta}}_{\dot{\alpha}}\nabla_{\alpha i}\nabla^{\gamma}{}_{\dot{\gamma}}W_{\beta\gamma}K^{\beta\dot{\gamma}}. \tag{104e}$$

We note that the conformal superspace algebra is expressed in terms of a single superfield $W_{\alpha\beta}=W_{(\alpha\beta)}$, its conjugate $\bar{W}_{\dot{\alpha}\dot{\beta}}$, and their covariant derivatives. This superfield is an $\mathcal{N}=2$ extension of the Weyl tensor and is called the super-Weyl tensor. It proves to be a primary chiral superfield of dimension 1,

我们注意到共形超空间代数可以用单个超场 $W_{\alpha\beta}=W_{(\alpha\beta)}$、其共轭 $\bar{W}_{\dot{\alpha}\dot{\beta}}$ 以及它们的协变导数来表示。该超场是魏尔张量的 $\mathcal{N}=2$ 推广，被称为超魏尔张量。它是维数为 1 的基本手征超场，

$$K^C W_{\alpha\beta}=0,\ \bar{\nabla}^{\dot{\gamma}}_k W_{\alpha\beta}=0,\ \mathbb{D}W_{\alpha\beta}=W_{\alpha\beta},\ \mathbb{Y}W_{\alpha\beta}=-2W_{\alpha\beta}, \tag{105}$$

and it obeys the Bianchi identity

且满足比安基恒等式

$$\mathfrak{B}\ :=\ \nabla_{\alpha\beta}W^{\alpha\beta}=\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{W}_{\dot{\alpha}\dot{\beta}}=\bar{\mathfrak{B}}, \tag{106a}$$

$$\nabla_{\alpha\beta}\ :=\ \nabla^i_{(\alpha}\nabla_{\beta)i},\ \bar{\nabla}^{\dot{\alpha}\dot{\beta}}\ :=\ \bar{\nabla}^{(\dot{\alpha}}_i\bar{\nabla}^{\dot{\beta})i}. \tag{106b}$$

The real scalar superfield $\mathfrak{B}$ is the $\mathcal{N}=2$ supersymmetric generalization of the Bach tensor. This super-Bach multiplet proves to be primary, $K^A\mathfrak{B}=0$, carries weight 2, $\mathbb{D}\mathfrak{B}=2\mathfrak{B}$, and satisfies the conservation equation [104],

实标量超场 $\mathfrak{B}$ 是巴赫张量的 $\mathcal{N}=2$ 超对称推广。这个超巴赫多重态是基本的，为 $K^A\mathfrak{B}=0$ ，携带权 2, $\mathbb{D}\mathfrak{B}=2\mathfrak{B}$ ，并满足守恒方程 [104],

$$\nabla^{ij}\mathfrak{B}=0\ \Leftrightarrow\ \bar{\nabla}_{ij}\mathfrak{B}=0, \tag{107a}$$

$$\nabla^{ij}\ :=\ \nabla^{\alpha(i}\nabla^{j)}_{\alpha},\ \bar{\nabla}_{ij}\ :=\ \bar{\nabla}_{\dot{\alpha}(i}\bar{\nabla}^{\dot{\alpha}}_{j)}. \tag{107b}$$

The structure of the conformal superspace algebra leads to highly nontrivial implications. In particular, Eq. (90c) implies that primary covariantly chiral superfields, $\bar{\nabla}^{\dot{\beta}}_j U=0$, can carry neither isospinor nor dotted spinor indices. Given such a superfield, $\phi_{\alpha(n)}\ :=\ \phi_{\alpha_1\ldots\alpha_n}=\phi_{(\alpha_1\ldots\alpha_n)}$, Eq. (90c) further implies that the $\mathrm{U}(1)_R$ charge of $\phi_{\alpha(n)}$ is determined in terms of its dimension,

共形超空间代数的结构具有高度非平庸的推论。特别地，式 (90c) 表明，基本协变手征超场 $\bar{\nabla}^{\dot{\beta}}_j U = 0$ 既不携带同旋量也不携带点旋量指标。对于这样的超场 $\phi_{\alpha(n)} := \phi_{\alpha_1 \ldots \alpha_n} = \phi_{(\alpha_1 \ldots \alpha_n)}$，式 (90c) 进一步说明，$\phi_{\alpha(n)}$ 的 $\mathrm{U}(1)_R$ 荷由其维数决定，

$$K^B \phi_{\alpha(n)} = 0, \ \bar{\nabla}^{\dot{\beta}}_j \phi_{\alpha(n)} = 0 \ \mathbb{D}\phi_{\alpha(n)} = w\phi_{\alpha(n)}, \ \mathbb{Y}\phi_{\alpha(n)} = -2w\phi_{\alpha(n)}$$

(108)

and thus $c = -2w$.

因此有 $c = -2w$ 。

There is a regular procedure to construct primary chiral multiplets and their conjugate antichiral ones. It is based on the use of operators

存在一套正则方法来构造基本手征多重态及其共轭反手征多重态，这套方法基于以下算符:

$$\nabla^4 := \frac{1}{48}\nabla^{ij}\nabla_{ij} = -\frac{1}{48}\nabla^{\alpha\beta}\nabla_{\alpha\beta}, \ \bar{\nabla}^4 := \frac{1}{48}\bar{\nabla}^{ij}\bar{\nabla}_{ij} = -\frac{1}{48}\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{\nabla}_{\dot{\alpha}\dot{\beta}}. \tag{109}$$

Let us consider a rank- n spinor superfield $\psi_{\alpha(n)}$ that is $\mathrm{SU}(2)_R$ neutral and has the following superconformal properties:

考虑一个秩- n 旋量超场 $\psi_{\alpha(n)}$，它是 $\mathrm{SU}(2)_R$ 中性的，具有如下超共形性质:

$$K^B \psi_{\alpha(n)} = 0, \ \mathbb{D}\psi_{\alpha(n)} = (w-2)\,\psi_{\alpha(n)}, \ \mathbb{Y}\psi_{\alpha(n)} = 2\,(2-w)\,\psi_{\alpha(n)}. \tag{110}$$

Then its descendant

那么它的 descendant(后代场)

$$\phi_{\alpha(n)} = \bar{\nabla}^4 \psi_{\alpha(n)} \tag{111}$$

is a primary covariantly chiral superfield of the type (108).

是一个满足 (108) 形式的基本协变手征超场。

Covariant Projective Multiplets

协变投影多重态

The concept of rigid superconformal projective multiplets, which was reviewed in section "Superconformal Primary Multiplets," naturally extends to conformal superspace. The operators (34) are replaced with

在“超共形原初多重态”小节回顾的刚性超共形投影多重态概念，可以自然推广到共形超空间。算符 (34) 被替换为

$$\nabla_{\alpha}^{(1)} = v_i \nabla_{\alpha}^{i}, \ \bar{\nabla}_{\dot{\alpha}}^{(1)} = v_i \bar{\nabla}_{\dot{\alpha}}^{i}, \tag{112}$$

which strictly anti-commute with each other due to (102). We recall that the rigid superconformal projective multiplet $Q^{(n)}(z,v)$ is defined by the relations (43), of which the conditions (43a) and (43b) trivially extend to conformal superspace,

由式 (102) 可知它们彼此严格反对易。我们回顾，刚性超共形投影多重态 $Q^{(n)}(z,v)$ 由关系 (43) 定义，其中条件 (43a) 和 (43b) 可以平凡推广到共形超空间，

$$K^A Q^{(n)} = 0, \ \nabla_{\alpha}^{(1)} Q^{(n)} = 0, \ \bar{\nabla}_{\dot{\alpha}}^{(1)} Q^{(n)} = 0, \tag{113a}$$

$$Q^{(n)}(z, \mathfrak{c}v) = \mathfrak{c}^n Q^{(n)}(z,v), \ \mathfrak{c} \in \mathbb{C}^*, \tag{113b}$$

while the rigid superconformal transformation law (43c) is replaced with

而刚性超共形变换规律 (43c) 被替换为

$$\delta_{\mathcal{K}} Q^{(n)} = \left(\xi^A \nabla_A + \Lambda^{ij} J_{ij} + \sum \mathbb{D}\right) Q^{(n)}, \tag{114a}$$

$$\Lambda^{ij} J_{ij} Q^{(n)} = -\left(\Lambda^{(2)} \partial^{(-2)} - n\Lambda^{(0)}\right) Q^{(n)}. \tag{114b}$$

Making use of the graded commutation relations (90b) and (90c) uniquely fixes the dimension of $Q^{(n)}$

利用阶化对易关系 (90b) 和 (90c) 可以唯一确定 $Q^{(n)}$ 的量纲

$$\mathbb{D} Q^{(n)} = n Q^{(n)}. \tag{114c}$$

We now list some projective multiplets that can be used to describe superfield dynamical variables. A complex $\mathcal{O}(m)$ multiplet, with $m = 1, 2, \ldots$, is described by a weight- m projective superfield $H^{(m)}(v)$ of the form:

我们现在列出一些可用于描述超场动力学变量的投影多重态。一个复 $\mathcal{O}(m)$ 多重态，带 $m = 1, 2, \ldots$ ，由权重为 m 的投影超场 $H^{(m)}(v)$ 描述，形式如下:

$$H^{(m)}(v) = v_{i_1} \ldots v_{i_m} H^{i_1 \cdots i_m}. \tag{115a}$$

The analyticity constraint (43a) is equivalent to

解析性约束 (43a) 等价于

$$\nabla_\alpha^{(i_1} H^{i_2 \cdots i_{m+1})} = 0, \ \bar{\nabla}_{\dot{\alpha}}^{(i_1} H^{i_2 \cdots i_{m+1})} = 0. \tag{115b}$$

If m is even, $m = 2n$, we can define a real $\mathcal{O}(2n)$ multiplet obeying the reality condition $\breve{H}^{(2n)} = H^{(2n)}$, or equivalently

若 m 是偶，$m = 2n$ ，我们可以定义满足实条件 $\breve{H}^{(2n)} = H^{(2n)}$ 的实 $\mathcal{O}(2n)$ 多重态，等价于

$$\overline{H^{i_1 \cdots i_{2n}}} = H_{i_1 \ldots i_{2n}} = \varepsilon_{i_1 j_1} \cdots \varepsilon_{i_{2n} j_{2n}} H^{j_1 \cdots j_{2n}}. \tag{116}$$

For $n > 1$, the real $\mathcal{O}(2n)$ multiplet can be used to describe an off-shell (neutral) hypermultiplet.

对于 $n > 1$ ，实 $\mathcal{O}(2n)$ 多重态可用于描述脱壳 (中性) 超多重态。

There is a simple construction to generate covariant projective multiplets. It makes use of isotwistor superfields. By definition, a weight- n isotwistor superfield $U^{(n)}(z, v)$ is a primary tensor superfield (with suppressed Lorentz indices) that has the following properties: (i) it is neutral with respect to the group $\mathrm{U}(1)_R$; (ii) it is holomorphic with respect to the isospinor variables v^i on an open domain of $\mathbb{C}^2 \setminus \{0\}$; (iii) it is a homogeneous function of v^i of degree n ,

存在一种构造协变投影多重态的简单方法，它利用等扭量超场。根据定义，权重为 n 的等扭量超场 $U^{(n)}(z, v)$ 是原初张量超场 (洛伦兹指标已省略)，具有如下性质:(i) 它相对于群 $\mathrm{U}(1)_R$ 是中性的；(ii) 它在 $\mathbb{C}^2 \setminus \{0\}$ 的开域上关于等旋量变量 v^i 是全纯的；(iii) 它是关于 v^i 的 n 次齐次函数,

$$U^{(n)}(\mathfrak{c} v) = \mathfrak{c}^n U^{(n)}(v), \ \mathfrak{c} \in \mathbb{C} \setminus \{0\}; \tag{117a}$$

and (iv) it is characterized by the gauge transformation law

且 (iv) 它由规范变换规律表征

$$\delta_{\mathcal{K}} U^{(n)} = \left(\xi^A \nabla_A + \frac{1}{2} K^{ab} M_{ab} + \Lambda^{ij} J_{ij} + \sum \mathbb{D} \right) U^{(n)},$$

$$J_{ij} U^{(n)} = -\left(v_{(i} v_{j)} \partial^{(-2)} - \frac{n}{(v, u)} v_{(i} u_{j)} \right) U^{(n)}. \tag{117b}$$

It is clear that any weight- n projective multiplet is an isotwistor superfield, but not vice versa. The main property in the definition of isotwistor superfields is their transformation rules under $\mathrm{SU}(2)_R$. In principle, the definition could be extended to consider non-primary superfields.

显然，任意权重为 n 的投影多重态都是等扭量超场，反之不成立。等扭量超场定义的核心性质是它们在 $\mathrm{SU}(2)_R$ 下的变换规则。原则上，该定义可以推广到非原初超场。

Let $U^{(n-4)}$ be a Lorentz-scalar isotwistor superfield such that

设 $U^{(n-4)}$ 是洛伦兹标量等扭量超场，满足

$$\mathbb{D}U^{(n-4)} = (n-2)\, U^{(n-4)}. \tag{118}$$

Then the weight- n isotwistor superfield

则权重为 n 的等扭量超场

$$Q^{(n)} \;:=\; \nabla^{(4)} U^{(n-4)} \tag{119}$$

satisfies all the properties of a covariant projective multiplet given by Eqs. (113) and (114). Here we have introduced the operator

满足式 (113) 和 (114) 给出的协变投影多重态的全部性质。这里我们引入了算符

$$\nabla^{(4)} = \frac{1}{16}\nabla^{(2)}\overline{\nabla}^{(2)},\ \nabla^{(2)} = v_i v_j \nabla^{ij},\ \overline{\nabla}^{(2)} = v_i v_j \overline{\nabla}^{ij}. \tag{120}$$

Component Reduction and the Weyl Multiplet

分量约化与魏尔多重态

Within the superconformal tensor calculus, the standard Weyl multiplet of conformal supergravity is associated with the local off-shell gauging in spacetime of the superconformal group SU$(2,2\mid 2)$ [7, 14 − 17] ; see also [18, 19] for a review. This multiplet comprises 24 + 24 physical components described by a set of independent gauge fields: the vielbein ${e_m}^a$ and a dilatation connection b_m ; the gravitino $\left({\psi_m}_i^\alpha, {\overline{\psi}_m}{}^i_{\dot\alpha}\right)$, associated with the gauging of Q -supersymmetry; a $\mathrm{U}(1)_R$ gauge field A_m ; and $\mathrm{SU}(2)_R$ gauge fields ${\phi_m}^{ij} = {\phi_m}^{ji}$. The fields associated with the remaining generators of SU$(2,2\mid 2)$, specifically the Lorentz connections ${\omega_m}^{cd}$, S -supersymmetry connection $\left({\phi_m}{}^i_\alpha, {\overline{\phi}_m}{}_i^{\dot\alpha}\right)$, and the special conformal connection f_{ma} , are composite fields. To ensure that the local superconformal transformations of the standard Weyl multiplet close off-shell, it is necessary to add a set of covariant matter fields. These are an antisymmetric real tensor $T_{ab} = T_{ba} = T^+_{ab} + T^-_{ab}$, which decomposes into its imaginary (anti-)self-dual components $T^\pm_{ab}$, a real scalar field D , and the fermions $\left(\Sigma^{\alpha i}, \overline{\Sigma}_{\dot\alpha i}\right)$.

在超共形张量微积分中，共形超引力的标准魏尔多重态对应超共形群 SU$(2,2\mid 2)$ [7, 14 − 17] 在时空的离壳局部规范变换；综述见 [18, 19] 。该多重态包含 24 + 24 个由一组独立规范场描述的物理分量: 标架 ${e_m}^a$ 和膨胀联络 b_m ；对应 Q 超对称性规范变换的引力微子 $\left({\psi_m}_i^\alpha, {\overline{\psi}_m}{}^i_{\dot\alpha}\right)$ ；一个 $\mathrm{U}(1)_R$ 规范场 A_m ；以及 $\mathrm{SU}(2)_R$ 个规范场 ${\phi_m}^{ij} = {\phi_m}^{ji}$ 。与 SU$(2,2\mid 2)$ 其余生成元对应的场，即洛伦兹联络 ${\omega_m}^{cd}$ 、S 超对称性联络 $\left({\phi_m}{}^i_\alpha, {\overline{\phi}_m}{}_i^{\dot\alpha}\right)$ 和特殊共形联络 f_{ma} ，均为复合场。为保证标准魏尔多重态的局部超共形变换在离壳下封闭，必须引入一组协变物质场: 反对称实张量 $T_{ab} = T_{ba} = T^+_{ab} + T^-_{ab}$ (可分解为虚 (反) 自对偶分量 $T^\pm_{ab}$)、实标量场 D ，以及费米子 $\left(\Sigma^{\alpha i}, \overline{\Sigma}_{\dot\alpha i}\right)$ 。

As described in the previous section, conformal superspace provides an off-shell gauging of the superconformal group SU$(2,2\mid 2)$ in superspace rather than spacetime. Apart from the fact that Q -supersymmetry is geometrically realized on superfields in a superspace setting, the conformal superspace and component approaches are very similar. In fact, it is straightforward to reduce the results of "Conformal Superspace" from superspace to spacetime and obtain all the details of the standard Weyl multiplet [63].

如前一节所述, 共形超空间实现了超共形群 SU (2, 2 | 2) 在超空间 (而非时空) 中的离壳规范变换。除了在超空间框架中 Q 超对称性以几何方式实现于超场之上，共形超空间方法与分量方法非常相似。事实上，我们可以直接将《共形超空间》中的结果从超空间约化到时空，得到标准魏尔多重态的全部细节 [63]。

The identification of the component gauge fields of the standard Weyl multiplet is straightforward. The vielbein $(e_m{}^a)$ and gravitini $\left(\psi_{m}{}_i^\alpha, \bar{\psi}_{m}{}_{\dot\alpha}^i\right)$ appear as the $\theta = 0$ projections of the coefficients of $\mathrm{d}x^m$ in the supervielbein E^A one-form,

标准魏尔多重态分量规范场的识别十分直接: 标架 $(e_m{}^a)$ 和引力微子 $\left(\psi_{m}{}_i^\alpha, \bar{\psi}_{m}{}_{\dot\alpha}^i\right)$ 对应超标架一元形式 E^A 中 $\mathrm{d}x^m$ 系数的 $\theta = 0$ 投影,

$$e^a = \mathrm{d}x^m e_m{}^a = E^a \,\|,\ \psi_i^\alpha = \mathrm{d}x^m \psi_m{}_i^\alpha = 2E_i^\alpha \,\|,\ \bar{\psi}_{\dot\alpha}^i = \mathrm{d}x^m \bar{\psi}_m{}_{\dot\alpha}^i = 2E_{\dot\alpha}^i \,\| \,. \tag{121}$$

Here we have defined the double bar projection of a superform as $\Omega \,\| \equiv \Omega|_{\theta=\mathrm{d}\theta=0}$. On the other hand, a single bar next to a superfield denotes the usual bar projection $X| \equiv X|_{\theta=0}$. The remaining component one-forms are defined as

此处我们将超形式的双杠投影定义为 $\Omega \,\| \equiv \Omega|_{\theta=\mathrm{d}\theta=0}$ ，而超场旁的单杠则表示通常的杠投影 $X| \equiv X|_{\theta=0}$ ，其余分量一元形式定义为

$$A := \Phi \,\|,\ \phi^{kl} := \Theta^{kl} \,\|,\ b := B \,\|,\ \omega^{cd} := \Omega^{cd} \,\|, \tag{122}$$

$$\phi_\gamma^k := 2\mathfrak{F}_\gamma^k \,\|,\ \bar{\phi}_k^{\dot\gamma} := 2\bar{\mathfrak{F}}_k^{\dot\gamma} \,\|,\ f_c := \mathfrak{F}_c \,\| \,. \tag{123}$$

The covariant matter fields T_{ab}, D , and $\left(\sum^{\alpha i}, \bar{\sum}_{\dot\alpha i}\right)$ arise as some of the components of the multiplet described by the super-Weyl tensor $W_{ab} = (\sigma_{ab})^{\alpha\beta} W_{\alpha\beta} - (\tilde\sigma_{ab})^{\dot\alpha\dot\beta} \bar{W}_{\dot\alpha\dot\beta}$, which satisfies the constraints (105) and (106a). In particular, it holds that

协变物质场 T_{ab}, D 和 $\left(\sum^{\alpha i}, \bar{\sum}_{\dot\alpha i}\right)$ 是超外尔张量 $W_{ab} = (\sigma_{ab})^{\alpha\beta} W_{\alpha\beta} - (\tilde\sigma_{ab})^{\dot\alpha\dot\beta} \bar{W}_{\dot\alpha\dot\beta}$ 描述的多重态的部分分量，该张量满足约束条件 (105) 和 (106a)。特别地，有如下关系成立

$$T_{ab} := W_{ab}\Big|,\ D = \frac{1}{12}\nabla^{\alpha\beta} W_{\alpha\beta}\Big| = \frac{1}{12}\bar\nabla^{\dot\alpha\dot\beta}\bar{W}_{\dot\alpha\dot\beta}\,|,| \tag{124a}$$

$$\sum{}^{\alpha i} = \frac{1}{3}\nabla_\beta^i W^{\alpha\beta}\Big|,\ \bar{\sum}_{\dot\alpha i} = -\frac{1}{3}\bar\nabla_i^{\dot\beta}\bar{W}_{\dot\alpha\dot\beta}\Big|. \tag{124b}$$

The local superconformal transformations of the gauge fields listed above can be straightforwardly derived by taking the $\theta = 0$ projection of the superspace transformations (93). At the same time, the transformations of T_{ab}, D , and $\left(\sum^{\alpha i}, \bar{\sum}_{\dot\alpha i}\right)$ can be obtained by applying the transformation rule for covariant superfields, Eqs. (98) and (91), and the definition of the descendant fields in Eq. (124). The resulting transformation laws are given in [105].

上述规范场的局部超共形变换可以通过对超空间变换 (93) 取 $\theta = 0$ 投影直接导出。同时，T_{ab}, D 和 $\left(\sum^{\alpha i}, \overline{\sum}_{\dot\alpha i}\right)$ 的变换可以通过应用协变超场的变换规则 (即式 (98) 和式 (91))，以及后代场的定义 (即式 (124)) 得到。最终的变换律已给出在文献 [105] 中。

By taking the double bar projection of the superspace covariant derivative one-form ∇ , Eq. (88), one defines a component vector covariant derivative as follows:

通过对超空间协变导数一形式 ∇ (即式 (88)) 取双杠投影，可以定义分量向量协变导数如下:

$$D = e^a D_a \ := \ \nabla \, \| \tag{125a}$$

$$e_m{}^a D_a = \partial_m - \frac{1}{2}\psi_m{}_i^\alpha \nabla_\alpha^i \left| - \frac{1}{2}\overline{\psi}_m{}^i_{\dot\alpha} \overline{\nabla}_i^{\dot\alpha} \right| - \frac{1}{2}\omega_m{}^{cd} M_{cd} - \mathrm{i} A_m \mathbb{Y} - \phi_m{}^{kl} J_{kl}$$

$$-b_m \mathbb{D} - \frac{1}{2}\phi_m{}^i_\alpha S_i^\alpha - \frac{1}{2}\overline{\phi}_m{}_i^{\dot\alpha} \bar{S}_{\dot\alpha}^i - \mathfrak{f}_{mc} K^c. \tag{125b}$$

Provided we appropriately interpret the projected spinor covariant derivatives $\nabla_\alpha^i \mid$ and $\overline{\nabla}_i^{\dot\alpha} \mid$ as the generators of Q -supersymmetry (Given a covariant superfield U , and its lowest component $\mathcal{U} = U \mid$, one defines $Q_\alpha^i \mathcal{U} = \nabla_\alpha^i \left| \mathcal{U} \ := \ (\nabla_\alpha^i U) \right|$ and $\bar{Q}_i^{\dot\alpha} \mathcal{U} = \overline{\nabla}_i^{\dot\alpha} \left| \mathcal{U} \ := \ \left(\overline{\nabla}_i^{\dot\alpha} U\right) \right|$. The action of the other generators $X_{\underline{a}}$ on $\mathcal{U}$ is simply given by $X_{\underline{a}} \mathcal{U} \ := \ \left(X_{\underline{a}} U\right) \mid$.), D describes a gauging in spacetime of the superconformal group SU $(2, 2 \mid 2)$, precisely as in [7]. This means that local diffeomorphisms, and all other structure group transformations of the derivatives (125), including Q -supersymmetry, consistently descend from their corresponding rule in superspace. With this interpretation, the algebra of component covariant derivatives acting on a covariant field is also completely determined by the geometry of conformal superspace. All the component torsions and curvatures are simply the $\theta = 0$ projections of the superspace ones. The algebra of D_a is (All fields and curvatures introduced so far satisfy natural conjugation properties. We refer the reader to [63] and, in particular, [105] for results in our notation, with the only difference being that the field W_{ab} in [105] is denoted as T_{ab} here.)

只要我们将投影后的旋量协变导数 $\nabla_\alpha^i \mid$ 和 $\overline{\nabla}_i^{\dot\alpha} \mid$ 恰当解释为 Q 超对称的生成元 (给定协变超场 U 及其最低分量 $\mathcal{U} = U \mid$ ，即可定义 $Q_\alpha^i \mathcal{U} = \nabla_\alpha^i \left| \mathcal{U} \ := \ (\nabla_\alpha^i U) \right|$ 和 $\bar{Q}_i^{\dot\alpha} \mathcal{U} = \overline{\nabla}_i^{\dot\alpha} \left| \mathcal{U} \ := \ \left(\overline{\nabla}_i^{\dot\alpha} U\right) \right|$ 。其余生成元 $X_{\underline{a}}$ 对 $\mathcal{U}$ 的作用可直接由 $X_{\underline{a}} \mathcal{U} \ := \ \left(X_{\underline{a}} U\right) \mid$ 给出)，D 就描述了超共形群 SU $(2, 2 \mid 2)$ 在时空中的规范作用，这与文献 [7] 中的结论完全一致。这说明局部微分同胚以及导数 (125) 所有其他结构群变换 (包括 Q 超对称) 都能够自洽地从超空间中的对应规则约化而来。在此解释下，作用在协变场上的分量协变导数代数也完全由共形超空间的几何确定。所有分量挠率和曲率都恰好是超空间挠率和曲率的 $\theta = 0$ 投影。D_a 的代数为 (目前引入的所有场和曲率都满足自然共轭性质。读者可参考文献 [63]，特别是文献 [105] 中对应我们记号的结果，唯一区别是文献 [105] 中的场 W_{ab} 在本文中记为 T_{ab} 。)

$$[D_a, D_b] = -R(P)_{ab}{}^c D_c - R(Q)_{ab}{}_i^\alpha \nabla_\alpha^i \left| - R(\bar{Q})_{ab}{}^i_{\dot\alpha} \overline{\nabla}_i^{\dot\alpha} \right|$$

$$-\frac{1}{2}R(M)_{ab}{}^{cd}M_{cd}-R(\mathbb{D})_{ab}\mathbb{D}-\mathrm{i}R(Y)_{ab}\mathbb{Y}-R(J)_{ab}{}^{kl}J_{kl}$$

$$-R(S)_{ab}{}_{\alpha}^{i}S_{i}^{\alpha}-R(\bar{S})_{ab}{}_{i}^{\dot{\alpha}}\bar{S}_{\dot{\alpha}}^{i}-R(K)_{abc}K^{c}. \tag{126}$$

By using the commutator of two superspace vector derivatives ∇_a , see [63], one can readily obtain all the component curvatures above. These prove to be determined by the lowest component of the super-Weyl tensor W_{ab} and its descendants. We do not present the results here but stress that the conformal superspace geometry implies the following conditions on the component superconformal curvatures:

利用两个超空间向量导数 ∇_a 的对易关系 (参见文献 [63])，可以很容易得到上述所有分量曲率。这些分量曲率都由超外尔张量 W_{ab} 的最低分量及其后代场决定。我们不在此列出结果，但需要强调共形超空间几何对分量超共形曲率施加了如下条件:

$$R(P)_{ab}{}^{c}=0, \tag{127a}$$

$$R(Q)_{ab}{}_{j}^{\beta}(\sigma^{b})_{\beta\dot{\alpha}}=-\frac{3}{4}\sum\nolimits_{j}^{\beta}(\sigma_{a})_{\beta\dot{\alpha}},\ R(\bar{Q})_{ab\dot{\beta}}(\tilde{\sigma}^{b})^{\dot{\beta}\alpha}=\frac{3}{4}\overline{\sum}{}_{\dot{\beta}}^{j}(\tilde{\sigma}_{a})^{\dot{\beta}\alpha}, \tag{127b}$$

$$R(M)^{c}{}_{acb}=R(\mathbb{D})_{ab}+3\eta_{ab}D-\eta^{cd}T_{ac}^{-}T_{bd}^{+}. \tag{127c}$$

These are the conventional constraints that render the connections $\omega_m{}^{cd}$, $\left(\phi_{m}{}_{\alpha}^{i},\overline{\phi}_{m}{}_{i}^{\dot{\alpha}}\right)$, and f_{ma} composite. We refer the reader to [63, 105] for the expressions of the composite connections and the superconformal curvatures expressed in terms of the independent physical fields of the standard Weyl multiplet. Note that the conventional constraints (127) are not the same as the ones originally employed in [7]. This is not surprising since there is large freedom in the choice of conventional constraints whenever it is necessary to add matter fields to achieve an off-shell representation. Different papers often make different choices. For example, the geometry of [7] is obtained through a shift of the special conformal connection $f_{ab}K^b$ proportional to DK_a ; see [63]. A particularly useful choice of constraints for calculations by using component fields is the ”traceless” one employed in [106]

这些是令联络 $\omega_m{}^{cd}$ 、$\left(\phi_{m}{}_{\alpha}^{i},\overline{\phi}_{m}{}_{i}^{\dot{\alpha}}\right)$ 和 f_{ma} 成为复合场的常规约束。关于复合联络以及用标准外尔多重态的独立物理场表示的超共形曲率的表达式，我们建议读者参阅 [63, 105] 。注意，此处的常规约束 (127) 与文献 [7] 最初采用的约束并不相同。这并不令人意外: 当需要引入物质场来实现离壳表示时，常规约束的选择有很大自由度。不同文献通常会做出不同选择。例如，文献 [7] 的几何是通过对特殊共形联络 $f_{ab}K^b$ 做一个正比于 DK_a 的平移得到的，参见文献 [63]。对于利用分量场进行计算而言，一个尤为实用的约束选择是文献 [106] 中采用的“无迹”约束

$$R(P)_{ab}{}^{c}=0,\ R(M)^{c}{}_{acb}=R(\mathbb{D})_{ab}, \tag{128a}$$

$$R(Q)_{ab}{}_{j}^{\beta}(\sigma^{b})_{\beta\dot{\alpha}}=0,\ R(\bar{Q})_{ab\dot{\beta}}{}^{j}(\tilde{\sigma}^{b})^{\dot{\beta}\alpha}=0. \tag{128b}$$

Other Superspace Formulations for Conformal Supergravity

共形超引力的其他超空间表述

As pointed out in section ”Introduction,” conformal superspace is not the only superspace setting to describe conformal supergravity. Here we consider two other covariant formulations that have found applications in recent years, specifically: (i) $U(2)$ superspace [26,59] and (ii) $SU(2)$ superspace [56,107]. They differ by their structure groups, which are $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(2)_R$ and $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)_R$, respectively. Below we describe the relevant ”degauging” procedures that lead to these geometries.

正如“引言”一节所指出，共形超空间并非描述共形超引力的唯一超空间框架。本文我们讨论近年来已得到应用的另外两种协变表述，具体为:(i) $U(2)$ 超空间 [26,59] 和 (ii) $SU(2)$ 超空间 [56,107]。二者的结构群不同，分别为 $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(2)_R$ 和 $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)_R$ 。下文我们介绍导出这些几何结构的相关“退规范”过程。

U(2) Superspace

U(2) 超空间

According to (91), under an infinitesimal special superconformal gauge transformation $\mathcal{K} = \Lambda_B K^B$, the dilatation connection transforms as follows:

根据式 (91)，在无穷小微分特殊超共形规范变换 $\mathcal{K} = \Lambda_B K^B$ 下，伸缩联络的变换形式如下:

$$\delta_{\mathcal{K}} B_A = -2\Lambda_A. \tag{129}$$

Thus, it is possible to choose a gauge condition $B_A = 0$, which completely fixes the special superconformal gauge freedom (There is a class of residual gauge transformations preserving the gauge $B_A = 0$. These generate the super-Weyl transformations of $U(2)$ superspace; see the next subsection.). As a result, the corresponding connection is no longer required for the covariance of ∇_A under the residual gauge freedom and may be extracted from ∇_A ,

因此，可以选取规范条件 $B_A = 0$ 来完全固定特殊超共形规范自由度 (存在一类保留规范 $B_A = 0$ 的剩余规范变换，它们生成 $U(2)$ 超空间的超外尔变换，参见下一小节)。由此，剩余规范自由度下 ∇_A 的协变性不再需要该联络，可将其从 ∇_A 中提出，

$$\nabla_A = \mathfrak{D}_A - \mathfrak{F}_{AB} K^B. \tag{130}$$

Here the operator $\mathfrak{D}_A$ involves only the Lorentz and $\mathrm{U}(2)_R$ connections

此处算符 $\mathfrak{D}_A$ 仅涉及洛伦兹联络和 $\mathrm{U}(2)_R$ 联络

$$\mathfrak{D}_A = E_A - \frac{1}{2}\Omega_A{}^{bc} M_{bc} - \Phi_A{}^{kl} J_{kl} - \mathrm{i}\Phi_A \mathbb{Y}. \tag{131}$$

It obeys the graded commutation relations

它满足阶化对易关系

$$[\mathfrak{D}_A, \mathfrak{D}_B\} = -\mathfrak{T}_{AB}{}^C \mathfrak{D}_C - \frac{1}{2}\mathfrak{R}_{AB}{}^{cd} M_{cd} - \mathfrak{R}_{AB}{}^{kl} J_{kl} - \mathrm{i}\mathfrak{R}_{AB}\mathbb{Y}. \tag{132}$$

The next step is to relate the special superconformal connection $\mathfrak{F}_{AB}$ to the torsion tensor of $U(2)$ superspace. To do this, one can make use of the relation

下一步是将特殊超共形联络 $\mathfrak{F}_{AB}$ 与 $U(2)$ 超空间的挠率张量联系起来。为此，可以利用如下关系

$$[\nabla_A, \nabla_B\} = [\mathfrak{D}_A, \mathfrak{D}_B\} - \left(\mathfrak{D}_A \mathfrak{F}_{BC} - (-1)^{AB} \mathfrak{D}_B \mathfrak{F}_{AC}\right) K^C - \mathfrak{F}_{AC}\left[K^C, \nabla_B\right\}$$

$$+(-1)^{AB}\mathfrak{F}_{BC}\left[K^C, \nabla_A\right\} + (-1)^{BC}\mathfrak{F}_{AC}\mathfrak{F}_{BD}\left[K^D, K^C\right\}. \tag{133}$$

In conjunction with (104), this relation leads to a set on consistency conditions that are equivalent to the Bianchi identities of $U(2)$ superspace [26]. Their solution expresses the components of $\mathfrak{F}_{AB}$ in terms of the torsion tensor of $U(2)$ superspace and completely determines the geometry of the $\mathfrak{D}_A$ derivatives [63]. Here we will present results only up to mass dimension-3/2. The outcome of the analysis is as follows:

结合式 (104)，该关系给出了一组与 $U(2)$ 超空间的比安基恒等式等价的相容性条件 [26]。它们的解用 $U(2)$ 超空间的挠率张量表示了 $\mathfrak{F}_{AB}$ 的分量，并完全确定了 $\mathfrak{D}_A$ 导数的几何结构 [63]。本文在此仅给出质量维数不超过 3/2 的结果。分析结论如下:

$$\mathfrak{F}^i_{\alpha\beta} = -\frac{1}{2}\varepsilon_{\alpha\beta} S^{ij} + \frac{1}{2}\varepsilon^{ij} Y_{\alpha\beta}, \tag{134a}$$

$$\mathfrak{F}_i^{\dot\alpha}{}_j^{\dot\beta} = -\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar{S}_{ij} + \frac{1}{2}\varepsilon_{ij}\bar{Y}^{\dot\alpha\dot\beta}, \tag{134b}$$

$$\mathfrak{F}^i_\alpha{}_j^{\dot\beta} = -\mathfrak{F}_j^{\dot\beta}{}^i_\alpha = -\delta^i_j G_\alpha{}^{\dot\beta} - \mathrm{i} G_\alpha{}^{\dot\beta}{}_j{}^i, \tag{134c}$$

$$\mathfrak{F}^i_{\alpha b} = -\frac{1}{2}(\tilde\sigma_b)^{\dot\beta\beta}\left\{\frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar{\mathfrak{D}}^{\dot\gamma i}\bar{W}_{\dot\beta\dot\gamma} - \frac{1}{6}\varepsilon_{\alpha\beta}\mathfrak{D}^\gamma_j G_{\gamma\dot\beta}{}^{ij} + \frac{\mathrm{i}}{12}\varepsilon_{\alpha\beta}\bar{\mathfrak{D}}_{\dot\beta j} S^{ij}\right.$$

$$\left. - \frac{\mathrm{i}}{4}\bar{\mathfrak{D}}^i_{\dot\beta} Y_{\alpha\beta} + \frac{1}{3}\mathfrak{D}_{(\alpha j}G_{\beta)\dot\beta}{}^{ij}\right\}, \tag{134d}$$

$$\mathfrak{F}_i^{\dot\alpha}{}_b = -\frac{1}{2}(\sigma_b)_{\beta\dot\beta}\left\{\frac{\mathrm{i}}{4}\varepsilon^{\dot\alpha\dot\beta}\mathfrak{D}_{\gamma i}W^{\beta\gamma} + \frac{1}{6}\varepsilon^{\dot\alpha\dot\beta}\bar{\mathfrak{D}}^j_{\dot\gamma}G^{\beta\dot\gamma}{}_{ij} + \frac{\mathrm{i}}{12}\varepsilon^{\dot\alpha\dot\beta}\mathfrak{D}^{\beta j}\bar{S}_{ij}\right.$$

$$\left. - \frac{\mathrm{i}}{4}\mathfrak{D}^\beta_i \bar{Y}^{\dot\alpha\dot\beta} - \frac{1}{3}\bar{\mathfrak{D}}^{(\dot\alpha j}G^{\beta\dot\beta)}{}_{ij}\right\} \tag{134e}$$

$$\mathfrak{F}_a{}^j_\beta = -\frac{1}{2}(\tilde\sigma_a)^{\dot\alpha\alpha}\left\{-\frac{\mathrm{i}}{12}\varepsilon_{\alpha\beta}\bar{\mathfrak{D}}_{\dot\beta j}S^{kj} - \frac{\mathrm{i}}{4}\bar{\mathfrak{D}}^j_{\dot\alpha}Y_{\alpha\beta} + \frac{1}{3}\mathfrak{D}_{\alpha k}G_{\beta\dot\alpha}{}^{jk}\right\}, \tag{134f}$$

$$\mathfrak{F}_a{}_j^{\dot\beta} = -\frac{1}{2}(\sigma_b)_{\alpha\dot\alpha}\left\{-\frac{\mathrm{i}}{12}\varepsilon^{\dot\alpha\dot\beta}\mathfrak{D}^{\beta j}\bar{S}_{kj} - \frac{\mathrm{i}}{4}\mathfrak{D}^\alpha_j\bar{Y}^{\dot\alpha\dot\beta} - \frac{1}{3}\bar{\mathfrak{D}}^{\dot\alpha k}G^{\alpha\dot\beta}{}_{jk}\right\}. \tag{134g}$$

The dimension-1 superfields have the following symmetry properties:

维数为 1 的超场具有如下对称性:

$$S^{ij}=S^{ji},\ Y_{\alpha\beta}=Y_{\beta\alpha},\ W_{\alpha\beta}=W_{\beta\alpha},\ G_{\alpha\dot{\alpha}}{}^{ij}=G_{\alpha\dot{\alpha}}{}^{ji}, \tag{135}$$

and the reality conditions

以及实条件

$$\overline{S^{ij}}=\bar{S}_{ij},\overline{W_{\alpha\beta}}=\bar{W}_{\dot{\alpha}\dot{\beta}},\overline{Y_{\alpha\beta}}=\bar{Y}_{\dot{\alpha}\dot{\beta}},\overline{G_{\beta\dot{\alpha}}}=G_{\alpha\dot{\beta}},\overline{G_{\beta\dot{\alpha}}{}^{ij}}=G_{\alpha\dot{\beta}ij}. \tag{136}$$

The $\mathrm{U}(1)_R$ charges of the complex fields are:

复场的 $\mathrm{U}(1)_R$ 荷为:

$$\mathbb{Y}S^{ij}=2S^{ij},\ \mathbb{Y}Y_{\alpha\beta}=2Y_{\alpha\beta},\ \mathbb{Y}W_{\alpha\beta}=-2W_{\alpha\beta}. \tag{137}$$

The algebra obeyed by $\mathfrak{D}_A$ takes the form:

$\mathfrak{D}_A$ 满足的代数形式为:

$$\left\{\mathfrak{D}_\alpha^i,\mathfrak{D}_\beta^j\right\}=4S^{ij}M_{\alpha\beta}+2\varepsilon_{\alpha\beta}\varepsilon^{ij}Y^{\gamma\delta}M_{\gamma\delta}+2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{W}_{\dot{\gamma}\dot{\delta}}\bar{M}^{\dot{\gamma}\dot{\delta}}$$

$$+2\varepsilon_{\alpha\beta}\varepsilon^{ij}S^{kl}J_{kl}+4Y_{\alpha\beta}J^{ij}, \tag{138a}$$

$$\left\{\mathfrak{D}_\alpha^i,\bar{\mathfrak{D}}_j^{\dot{\beta}}\right\}=-2\mathrm{i}\delta_j^i\mathfrak{D}_\alpha{}^{\dot{\beta}}+4\left(\delta_j^iG^{\gamma\dot{\beta}}+\mathrm{i}G^{\gamma\dot{\beta}i}{}_j\right)M_{\alpha\gamma}+4\left(\delta_j^iG_{\alpha\dot{\gamma}}+\mathrm{i}G_{\alpha\dot{\gamma}}{}^i{}_j\right)\bar{M}^{\dot{\beta}\dot{\gamma}}$$

$$+8G_\alpha{}^{\dot{\beta}}J^i{}_j-4\mathrm{i}\delta_j^iG_\alpha{}^{\dot{\beta}kl}J_{kl}-2\left(\delta_j^iG_\alpha{}^{\dot{\beta}}+\mathrm{i}G_\alpha{}^{\dot{\beta}i}{}_j\right)\mathbb{Y}, \tag{138b}$$

$$\left[\mathfrak{D}_a,\mathfrak{D}_\beta^j\right]=-\mathrm{i}(\tilde{\sigma}_a)^{\dot{\alpha}\gamma}\left(\delta_k^jG_{\beta\dot{\alpha}}+\mathrm{i}G_{\beta\dot{\alpha}}{}^j{}_k\right)\mathfrak{D}_\gamma^k$$

$$+\frac{\mathrm{i}}{2}\left((\sigma_a)_{\beta\dot{\gamma}}S^{jk}-\varepsilon^{jk}(\sigma_a)_\beta{}^{\dot{\delta}}\bar{W}_{\dot{\delta}\dot{\gamma}}-\varepsilon^{jk}(\sigma_a)^\alpha{}_{\dot{\gamma}}Y_{\alpha\beta}\right)\bar{\mathfrak{D}}_k^{\dot{\gamma}}$$

$$-\frac{1}{2}\mathfrak{R}_{a\beta}{}^{j\,cd}M_{cd}-\mathfrak{R}_{a\beta}{}^{j\,kl}J_{kl}-\mathrm{i}\mathfrak{R}_{a\beta}{}^j\mathbb{Y} \tag{138c}$$

The dimension-3/2 components of the curvature appearing in (138c) are

式 (138c) 中出现的曲率维数为 3/2 的分量为

$$\mathfrak{R}_{a\beta cd}^{\ j}=-\mathrm{i}(\sigma_d)_\beta{}^{\dot{\delta}}\mathfrak{T}_{ac\dot{\delta}}^{\ \ j}+\mathrm{i}(\sigma_a)_\beta{}^{\dot{\delta}}\mathfrak{T}_{cd\dot{\delta}}^{\ \ j}-\mathrm{i}(\sigma_c)_\beta{}^{\dot{\delta}}\mathfrak{T}_{da\dot{\delta}}^{\ \ j}, \tag{139a}$$

$$\mathfrak{R}_{a\beta}{}^{jkl} = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\Big\{\mathrm{i}\varepsilon^{j(k}\bar{\mathfrak{D}}^{l)}_{\dot{\alpha}}Y_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{j(k}\bar{\mathfrak{D}}^{\dot{\delta}l)}\bar{W}_{\dot{\alpha}\dot{\delta}} + \frac{\mathrm{i}}{3}\varepsilon_{\alpha\beta}\varepsilon^{j(k}\bar{\mathfrak{D}}_{\dot{\alpha}q}S^{l)q}$$

$$-\frac{4}{3}\varepsilon^{j(k}\mathfrak{D}_{(\alpha q}G_{\beta)\dot{\alpha}}{}^{l)q} - \frac{2}{3}\varepsilon_{\alpha\beta}\varepsilon^{j(k}\mathfrak{D}^{\delta}_{q}G_{\delta\dot{\alpha}}{}^{l)q}\Big\}, \tag{139b}$$

$$\mathfrak{R}_{a\beta}{}^{j} = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\Big\{\mathfrak{D}^{j}_{\beta}G_{\alpha\dot{\alpha}} - \frac{\mathrm{i}}{3}\mathfrak{D}_{(\alpha k}G_{\beta)\dot{\alpha}}{}^{jk} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\mathfrak{D}^{\gamma}_{k}G_{\gamma\dot{\alpha}}{}^{jk}\Big\}, \tag{139c}$$

together with their complex conjugates. The right-hand side of (139a) involves the dimension-3/2 components of the torsion, which take the form

以及它们的复共轭。式 (139a) 的右侧包含挠率维数为 3/2 的分量，其形式为

$$\mathfrak{T}^{k}_{ab\dot{\gamma}} \equiv (\sigma_{ab})^{\alpha\beta}\mathfrak{T}^{k}_{\alpha\beta\dot{\gamma}} - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\mathfrak{T}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}, \tag{140a}$$

$$\mathfrak{T}^{k}_{\alpha\beta\dot{\gamma}} = \frac{1}{4}\bar{\mathfrak{D}}^{k}_{\dot{\gamma}}Y_{\alpha\beta} - \frac{\mathrm{i}}{3}\mathfrak{D}^{l}_{(\alpha}G_{\beta)\dot{\gamma}}{}^{k}{}_{l}, \tag{140b}$$

$$\mathfrak{T}^{k}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \frac{1}{4}\bar{\mathfrak{D}}^{k}_{\dot{\gamma}}\bar{W}_{\dot{\alpha}\dot{\beta}} + \frac{1}{6}\varepsilon_{\dot{\gamma}(\dot{\alpha}}\bar{\mathfrak{D}}_{\dot{\beta})l}S^{kl} + \frac{\mathrm{i}}{3}\varepsilon_{\dot{\gamma}(\dot{\alpha}}\mathfrak{D}^{\delta}_{q}G_{\delta\dot{\beta})}{}^{kq}. \tag{140c}$$

The consistency conditions arising from solving (133) and the constraints satisfied by $W_{\alpha\beta}$ in conformal superspace lead to the following set of dimension-3/2 Bianchi identities:

通过求解方程 (133) 得到的相容性条件，以及共形超空间中 $W_{\alpha\beta}$ 满足的约束，可推导出如下一组 3/2 维比安基恒等式:

$$\mathfrak{D}^{(i}_{\alpha}S^{jk)} = 0, \tag{141a}$$

$$\mathfrak{D}^{i}_{\alpha}\bar{W}_{\dot{\beta}\dot{\gamma}} = 0, \tag{141b}$$

$$\mathfrak{D}^{i}_{(\alpha}Y_{\beta\gamma)} = 0, \tag{141c}$$

$$\mathfrak{D}^{(i}_{(\alpha)}G_{\beta)\dot{\beta}}{}^{jk)} = 0, \tag{141d}$$

$$\bar{\mathfrak{D}}^{(i)}_{\dot{\alpha}}S^{jk)} = \mathrm{i}\mathfrak{D}^{\beta\left(i}G_{\beta\dot{\alpha}}{}^{jk\right)}, \tag{141e}$$

$$\mathfrak{D}^{i}_{\alpha}S_{ij} = -\mathfrak{D}^{\beta}_{j}Y_{\beta\alpha}, \tag{141f}$$

$$\mathfrak{D}^{i}_{\alpha}G_{\beta\dot{\beta}} = -\frac{1}{4}\bar{\mathfrak{D}}^{i}_{\dot{\beta}}Y_{\alpha\beta} + \frac{1}{12}\varepsilon_{\alpha\beta}\bar{\mathfrak{D}}_{\dot{\beta}j}S^{ij} - \frac{1}{4}\varepsilon_{\alpha\beta}\bar{\mathfrak{D}}^{\dot{\gamma}i}\bar{W}_{\dot{\gamma}\dot{\beta}}$$

$$-\frac{\mathrm{i}}{3}\varepsilon_{\alpha\beta}\mathfrak{D}^{\gamma}_{j}G_{\gamma\dot{\beta}}{}^{ij}, \tag{141g}$$

and the dimension-2 constraint

以及维数为 2 的约束

$$\left(\mathfrak{D}_{\alpha\beta}-4Y_{\alpha\beta}\right)W^{\alpha\beta}=\left(\overline{\mathfrak{D}}^{\dot{\alpha}\dot{\beta}}-4\bar{Y}^{\dot{\alpha}\dot{\beta}}\right)\bar{W}_{\dot{\alpha}\dot{\beta}}. \tag{142}$$

Here we have made the definitions

此处我们给出定义:

$$\mathfrak{D}_{\alpha\beta}=\mathfrak{D}^{i}_{(\alpha}\mathfrak{D}_{\beta)i},\ \overline{\mathfrak{D}}_{\dot{\alpha}\dot{\beta}}=\overline{\mathfrak{D}}_{i}^{(\dot{\alpha}}\overline{\mathfrak{D}}^{\dot{\beta})i}, \tag{143}$$

and it is useful to also define

同时，额外给出下述定义也十分有用

$$\mathfrak{D}^{ij}=\mathfrak{D}^{\alpha(i}\mathfrak{D}^{j)}_{\alpha},\ \overline{\mathfrak{D}}_{ij}=\overline{\mathfrak{D}}_{\dot{\alpha}(i}\overline{\mathfrak{D}}^{\dot{\alpha}}_{j)}. \tag{144}$$

In closing, we note that, upon degauging, relation (111) takes the form [58,108]

最后我们指出，退规范后，关系式 (111) 会变为如下形式 [58,108]

$$\begin{aligned}\phi_{\alpha(n)}&=\left(\frac{1}{96}\bar{\mathfrak{D}}^{ij}\overline{\mathfrak{D}}_{ij}-\frac{1}{96}\overline{\mathfrak{D}}_{\dot{\alpha}\dot{\beta}}\overline{\mathfrak{D}}^{\dot{\alpha}\dot{\beta}}+\frac{1}{6}\bar{S}^{ij}\overline{\mathfrak{D}}_{ij}+\frac{1}{6}\bar{Y}_{\dot{\alpha}\dot{\beta}}\overline{\mathfrak{D}}^{\dot{\alpha}\dot{\beta}}\right)\psi_{\alpha(n)}\\&\equiv\overline{\Delta}\psi_{\alpha(n)}\end{aligned} \tag{145}$$

The Super-Weyl Transformations of U(2) Superspace

U(2) 超空间的超外尔变换

In the previous subsection we made use of the special conformal gauge freedom to degauge from conformal to $U(2)$ superspace. The goal of this subsection is to show that residual dilatation symmetry manifests in the latter as super-Weyl transformations.

在上一小节中，我们利用特殊共形规范自由度从共形超空间退规范到了 $U(2)$ 超空间。本小节的目标是说明，剩余标度对称性在后者中体现为超外尔变换。

To preserve the gauge $B_A=0$, every local dilatation transformation with parameter $\sum$ should be accompanied by a compensating special conformal one

为了保留 $B_A=0$ 规范，每个参数为 $\sum$ 的局域标度变换都必须伴随一个补偿性的特殊共形变换

$$\mathcal{K}\left(\sum\right)=\Lambda_B\left(\sum\right)K^B+\sum\mathbb{D}\ \Rightarrow\ \delta_{\mathcal{K}(\sum)}B_A=0. \tag{146}$$

We then arrive at the following constraints:

我们于是得到如下约束条件:

$$\Lambda_A\left(\sum\right)=\frac{1}{2}\nabla_A\sum. \tag{147}$$

As a result, we define the following transformation:

据此，我们定义如下变换:

$$\delta_{\sum}\nabla_A=\delta_{\sum}\mathfrak{D}_A-\delta_{\sum}\mathfrak{F}_{AB}K^B=\left[\mathcal{K}\left(\sum\right),\nabla_A\right]. \tag{148}$$

By making use of (134), one can obtain the following transformation laws for the U(2) superspace covariant derivatives:

利用式 (134)，可以得到 U(2) 超空间协变导数的下述变换规律:

$$\delta_{\sum}\mathfrak{D}_\alpha^i=\frac{1}{2}\sum\mathfrak{D}_\alpha^i+2\left(\mathfrak{D}^{\gamma i}\sum\right)M_{\gamma\alpha}-2\left(\mathfrak{D}_{\alpha k}\sum\right)J^{ki}-\frac{1}{2}\left(\mathfrak{D}_\alpha^i\sum\right)\mathbb{Y}, \tag{149a}$$

$$\delta_{\sum}\overline{\mathfrak{D}}_{\dot\alpha i}=\frac{1}{2}\sum\overline{\mathfrak{D}}_{\dot\alpha i}+2\left(\overline{\mathfrak{D}}_i^{\dot\gamma}\sum\right)\bar{M}_{\dot\gamma\dot\alpha}+2\left(\overline{\mathfrak{D}}_{\dot\alpha}^k\sum\right)J_{ki}+\frac{1}{2}\left(\overline{\mathfrak{D}}_{\dot\alpha i}\sum\right)\mathbb{Y}, \tag{149b}$$

$$\delta_{\sum}\mathfrak{D}_{\alpha\dot\alpha}=\sum\mathfrak{D}_{\alpha\dot\alpha}+\mathrm{i}\left(\overline{\mathfrak{D}}_{\dot\alpha k}\sum\right)\mathfrak{D}_\alpha^k+\mathrm{i}\left(\mathfrak{D}_\alpha^k\sum\right)\overline{\mathfrak{D}}_{\dot\alpha k}$$

$$+\left(\mathfrak{D}^\gamma{}_{\dot\alpha}\sum\right)M_{\gamma\alpha}+\left(\mathfrak{D}_\alpha{}^{\dot\gamma}\sum\right)\bar{M}_{\dot\gamma\dot\alpha}. \tag{149c}$$

The dimension-1 components of the torsion transform as

1 次挠率分量的变换为

$$\delta_{\sum}W_{\alpha\beta}=\sum W_{\alpha\beta}, \tag{150a}$$

$$\delta_{\sum}Y_{\alpha\beta}=\sum Y_{\alpha\beta}-\frac{1}{2}\mathfrak{D}_{\alpha\beta}\sum, \tag{150b}$$

$$\delta_{\sum}S_{ij}=\sum S_{ij}-\frac{1}{2}\mathfrak{D}_{ij}\sum \tag{150c}$$

$$\delta_{\sum}G_{\alpha\dot\alpha}=\sum G_{\alpha\dot\alpha}-\frac{1}{8}\left[\mathfrak{D}_\alpha^k,\overline{\mathfrak{D}}_{\dot\alpha k}\right]\sum, \tag{150d}$$

$$\delta_{\sum}G_{\alpha\dot\alpha}{}^{ij}=\sum G_{\alpha\dot\alpha}{}^{ij}+\frac{\mathrm{i}}{4}\left[\mathfrak{D}_\alpha^{(i},\overline{\mathfrak{D}}_{\dot\alpha}^{j)}\right]\sum. \tag{150e}$$

SU(2) Superspace

SU(2) 超空间

It can be proven that the torsion ${G_{\alpha\dot{\alpha}}}^{ij}$ of U (2) superspace is a pure gauge degree of freedom [26, 59]. One can use super-Weyl gauge freedom (150e) to choose

可以证明，U (2) 超空间的挠率 ${G_{\alpha\dot{\alpha}}}^{ij}$ 是纯规范自由度 [26, 59]。我们可以利用超外尔规范自由度 (150e) 选取

$$
{G_{\alpha\dot{\beta}}}^{ij} = 0. \tag{151}
$$

In this gauge, it is natural to introduce new covariant derivatives $\mathcal{D}_A$ defined by

在此规范下，引入由下式定义的新协变导数 $\mathcal{D}_A$ 是自然的

$$
\mathcal{D}_\alpha^i = \mathfrak{D}_\alpha^i, \ \mathcal{D}_a = \mathfrak{D}_a - \mathrm{i} G_a \mathbb{Y}. \tag{152}
$$

Making use of (138), we find that they obey the graded commutation relations

利用 (138)，我们可得它们满足分次对易关系

$$
\left\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\right\} = 4S^{ij} M_{\alpha\beta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta} Y^{\gamma\delta} M_{\gamma\delta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta} \bar{W}^{\dot{\gamma}\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}}
$$

$$
+2\varepsilon_{\alpha\beta}\varepsilon^{ij} S^{kl} J_{kl} + 4Y_{\alpha\beta} J^{ij}, \tag{153a}
$$

$$
\left\{\mathcal{D}_\alpha^i, \overline{\mathcal{D}}_j^{\dot{\beta}}\right\} = -2\mathrm{i}\delta_j^i {(\sigma^c)_\alpha}^{\dot{\beta}} \mathcal{D}_c + 4\delta_j^i G^{\delta\dot{\beta}} M_{\alpha\delta} + 4\delta_j^i G_{\alpha\dot{\gamma}} \bar{M}^{\dot{\gamma}\dot{\beta}} + 8{G_\alpha}^{\dot{\beta}} {J^i}_j, \tag{153b}
$$

$$
\left[\mathcal{D}_a, \mathcal{D}_\beta^j\right] = \mathrm{i}{(\sigma_a)_{(\beta}}^{\dot{\beta}} G_{\gamma)\dot{\beta}} \mathcal{D}^{\gamma j}
$$

$$
+\frac{\mathrm{i}}{2}\left((\sigma_a)_{\beta\dot{\gamma}} S^{jk} - \varepsilon^{jk} {(\sigma_a)_\beta}^{\dot{\delta}} \bar{W}_{\dot{\delta}\dot{\gamma}} - \varepsilon^{jk} {(\sigma_a)^\alpha}_{\dot{\gamma}} Y_{\alpha\beta}\right) \overline{\mathcal{D}}_k^{\dot{\gamma}}
$$

$$
+\frac{\mathrm{i}}{2}\left((\tilde{\sigma}_a)^{\dot{\gamma}\gamma} \varepsilon^{j(k} \overline{\mathcal{D}}_{\dot{\gamma}}^{l)} Y_{\beta\gamma} - (\sigma_a)_{\beta\dot{\gamma}} \varepsilon^{j(k} \overline{\mathcal{D}}_{\dot{\delta}}^{l)} \bar{W}^{\dot{\gamma}\dot{\delta}} - \frac{1}{2} {(\sigma_a)_\beta}^{\dot{\gamma}} \overline{\mathcal{D}}_{\dot{\gamma}}^j S^{kl}\right) J_{kl}
$$

$$
+\frac{\mathrm{i}}{2}\left({(\sigma_a)_\beta}^{\dot{\delta}} \hat{\mathcal{T}}_{cd}{}_{\dot{\delta}}^j + {(\sigma_c)_\beta}^{\dot{\delta}} \hat{\mathcal{T}}_{ad}{}_{\dot{\delta}}^j - {(\sigma_d)_\beta}^{\dot{\delta}} \hat{\mathcal{T}}_{ac}{}_{\dot{\delta}}^j\right) M^{cd}, \tag{153c}
$$

where

其中

$$
\hat{\mathcal{T}}_{ab\dot{\gamma}}^k = -\frac{1}{4} (\sigma_{ab})^{\alpha\beta} \bar{\mathcal{D}}_{\dot{\gamma}}^k Y_{\alpha\beta} + \frac{1}{4} (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}_{\dot{\gamma}}^k \bar{W}_{\dot{\alpha}\dot{\beta}} - \frac{1}{6} (\tilde{\sigma}_{ab})_{\dot{\gamma}\dot{\delta}} \bar{\mathcal{D}}_l^{\dot{\delta}} S^{kl}. \tag{153d}
$$

The various torsion tensors in (153) obey the Bianchi identities (141) and (142) upon the replacement $\mathfrak{D}_A \to \mathcal{D}_A$ and imposing (151). By examining equations (153) we see that the $\mathrm{U}(1)_R$ curvature has been eliminated and therefore the corresponding connection is flat. Hence, by performing an appropriate local $\mathrm{U}(1)_R$ transformation, it may be gauged away

(153) 中的各类挠率张量在替换 $\mathfrak{D}_A \to \mathcal{D}_A$ 并施加 (151) 后满足比安基恒等式 (141) 和 (142)。考察方程 (153) 可知，$\mathrm{U}(1)_R$ 曲率已被消去，因此相应联络是平坦的。故通过做适当的局域 $\mathrm{U}(1)_R$ 变换，可将其规范掉

$$\Phi_A = 0\,. \tag{154}$$

As a result, the gauge group reduces to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)_R$ and the superspace geometry is the so-called SU (2) superspace of [56, 107] .

因此，规范群约化为 $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)_R$ ，该超空间几何就是所谓 [56, 107] 的 SU (2) 超空间。

It turns out that the gauge conditions (151) and (154) allow for residual super-Weyl transformations, which are described by a parameter $\sum$ constrained by

可以证明，规范条件 (151) 和 (154) 仍保留残余超外尔变换，这类变换由满足下式约束的参数 $\sum$ 描述:

$$\left[\mathfrak{D}_\alpha^{(i}, \bar{\mathfrak{D}}_{\dot\alpha}^{j)}\right] \sum = 0. \tag{155}$$

The general solution of this condition is [56]

该条件的通解为 [56]

$$\sum = \frac{1}{2}\left(\sigma + \bar\sigma\right),\ \bar{\mathfrak{D}}_i^{\dot\alpha}\sigma = 0,\ \mathbb{Y}\sigma = 0, \tag{156}$$

where the parameter σ is covariantly chiral, with zero $\mathrm{U}(1)_R$ charge, but otherwise arbitrary. To preserve the gauge condition $\Phi_A = 0$, every super-Weyl transformation, see (149a) and (149b), must be accompanied by the following compensating $\mathrm{U}(1)_R$ transformation:

其中参数 σ 是协变手征的，$\mathrm{U}(1)_R$ 荷为零，除此之外任意。为保持规范条件 $\Phi_A = 0$ ，每个超外尔变换 (见 (149a) 和 (149b)) 都必须伴随如下补偿性 $\mathrm{U}(1)_R$ 变换:

$$\delta\mathfrak{D}_A = [\mathrm{i}\rho\mathbb{Y}, \mathfrak{D}_A]\,,\ \rho = \frac{\mathrm{i}}{4}\left(\sigma - \bar\sigma\right). \tag{157}$$

As a result, the SU(2) geometry is left invariant by the following set of super-Weyl transformations [56]:

因此，SU(2) 几何在下述超外尔变换集合下保持不变 [56]:

$$\delta_\sigma\mathcal{D}_\alpha^i = \frac{1}{2}\bar\sigma\mathcal{D}_\alpha^i + \left(\mathcal{D}^{\gamma i}\sigma\right)M_{\gamma\alpha} - \left(\mathcal{D}_{\alpha k}\sigma\right)J^{ki}, \tag{158a}$$

$$\delta_\sigma \overline{\mathcal{D}}_{\dot{\alpha} i} = \frac{1}{2}\sigma \overline{\mathcal{D}}_{\dot{\alpha} i} + \left(\overline{\mathcal{D}}_i^{\dot{\gamma}} \overline{\sigma}\right) \overline{M}_{\dot{\gamma}\dot{\alpha}} + \left(\overline{\mathcal{D}}_{\dot{\alpha}}^k \overline{\sigma}\right) J_{ki}, \tag{158b}$$

$$\delta_\sigma \mathcal{D}_a = \frac{1}{2}\left(\sigma + \overline{\sigma}\right) \mathcal{D}_a + \frac{\mathrm{i}}{4} (\sigma_a)^{\alpha}{}_{\dot{\beta}} \left(\mathcal{D}_\alpha^k \sigma\right) \overline{\mathcal{D}}_k^{\dot{\beta}} + \frac{\mathrm{i}}{4} (\sigma_a)^{\alpha}{}_{\dot{\beta}} \left(\overline{\mathcal{D}}_k^{\dot{\beta}} \overline{\sigma}\right) \mathcal{D}_\alpha^k$$

$$-\frac{1}{2}\left(\mathcal{D}^b \left(\sigma + \overline{\sigma}\right)\right) M_{ab} \tag{158c}$$

$$\delta_\sigma S^{ij} = \overline{\sigma} S^{ij} - \frac{1}{4}\mathcal{D}^{ij}\sigma, \tag{158d}$$

$$\delta_\sigma Y_{\alpha\beta} = \overline{\sigma} Y_{\alpha\beta} - \frac{1}{4}\mathcal{D}_{\alpha\beta}\sigma, \tag{158e}$$

$$\delta_\sigma W_{\alpha\beta} = \sigma W_{\alpha\beta}, \tag{158f}$$

$$\delta_\sigma G_{\alpha\dot{\beta}} = \frac{1}{2}\left(\sigma + \overline{\sigma}\right) G_{\alpha\dot{\beta}} - \frac{\mathrm{i}}{4}\mathcal{D}_{\alpha\dot{\beta}}\left(\sigma - \overline{\sigma}\right). \tag{158g}$$

Here we have made use of the definitions

此处我们利用了定义

$$\mathcal{D}_{\alpha\beta} = \mathcal{D}^i_{(\alpha)\beta},\ \mathcal{D}^{ij} = \mathcal{D}^{\alpha(i)}\mathcal{D}^{j)}_\alpha, \tag{159}$$

and it is useful to also define

另外定义下述量是有用的

$$\overline{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} = \overline{\mathcal{D}}_i^{(\dot{\alpha})\overline{\mathcal{D}}\dot{\beta} i},\ \overline{\mathcal{D}}_{ij} = \overline{\mathcal{D}}_{\dot{\alpha}(i}\overline{\mathcal{D}}^{\dot{\alpha}}_{j)}. \tag{160}$$

Due to these transformations, SU(2) superspace provides a geometric description of the Weyl multiplet of $\mathcal{N} = 2$ conformal supergravity [56]. It should be emphasized that the algebra of covariant derivatives (153) was derived originally by Grimm [107]. However, no discussion of super-Weyl transformations was given in [107]

由于这些变换性质，SU(2) 超空间为 $\mathcal{N} = 2$ 共形超引力的外尔多重态提供了几何描述 [56]。需要强调的是，协变导数代数 (153) 最初由 Grimm 导出 [107]，但文献 [107] 未讨论超外尔变换

Let us fix a background curved superspace $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$. A supervector field $\xi = \xi^B E_B$ on this superspace is called conformal Killing if there exist a Lorentz parameter $K^{bc}[\xi]$, $\mathrm{SU}(2)_R$ parameter $\Lambda^{ij}[\xi]$, and a chiral super-Weyl parameter $\sigma[\xi]$ such that

我们固定一个背景弯曲超空间 $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$。该超空间上的超向量场 $\xi = \xi^B E_B$ 若满足下述条件则称为共形 Killing 场: 存在洛伦兹参数 $K^{bc}[\xi]$, $\mathrm{SU}(2)_R$ 、参数 $\Lambda^{ij}[\xi]$ 和手征超外尔参数 $\sigma[\xi]$，使得

$$\left[\xi^B \mathcal{D}_B + \frac{1}{2}K^{bc}[\xi] M_{bc} + \Lambda^{ij}[\xi] J_{ij}, \mathcal{D}_A\right] + \delta_{\sigma[\xi]}\mathcal{D}_A = 0. \tag{161}$$

In other words, the coordinate transformation generated by ξ is accompanied by certain Lorentz, $\mathrm{SU}(2)_R$, and super-Weyl transformations such that the superspace geometry does not change. It can be shown that Equation (161) uniquely determines the spinor components of $\xi^B = \left(\xi^b, \xi_j^\beta, \bar{\xi}_{\dot\beta}^j\right)$ and the parameters $K^{bc}[\xi], \Lambda^{ij}[\xi]$ and $\sigma[\xi]$ in terms of ξ^b , and the latter obeys the equation

换句话说, ξ 生成的坐标变换伴随特定洛伦兹变换、$\mathrm{SU}(2)_R$ 变换和超外尔变换, 使得超空间几何保持不变。可以证明，方程 (161) 由 ξ^b 唯一确定了 $\xi^B = \left(\xi^b, \xi_j^\beta, \bar{\xi}_{\dot\beta}^j\right)$ 的旋量分量以及参数 $K^{bc}[\xi], \Lambda^{ij}[\xi]$ 和 $\sigma[\xi]$，且 ξ^b 满足方程

$$\mathcal{D}^i_{(\alpha\dot\beta)\dot\beta} = 0 \Leftrightarrow \bar{\mathcal{D}}^i_{(\dot\alpha}\xi_{\beta\dot\beta)} = 0. \tag{162}$$

The set of all conformal Killing supervector fields on $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ constitutes the superconformal algebra of $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$. Given a super-Weyl invariant theory on $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ described by primary superfields U , its action is invariant under the superconformal transformations

$\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ 上所有共形 Killing 超向量场的集合构成了 $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ 的超共形代数。对于由主超场 U 描述的 $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ 上的超外尔不变理论，其作用量在超共形变换下保持不变

$$\delta_\xi U = \mathcal{K}[\xi] U$$

$$\mathcal{K}[\xi] = \xi^B \mathcal{D}_B + \frac{1}{2} K^{bc}[\xi] M_{bc} + \Lambda^{ij}[\xi] J_{ij} + p\sigma[\xi] + q\bar{\sigma}[\xi], \tag{163}$$

for an arbitrary conformal Killing supervector field ξ . In the case that $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ coincides with Minkowski superspace, $\left(\mathbb{M}^{4|8}, D\right)$, the superconformal Killing equation (161) is equivalent to (10) and the transformation law (163) to (29).

对任意共形 Killing 超向量场 ξ 成立。当 $\left(\mathcal{M}^{4|8}, \mathcal{D}\right)$ 为闵可夫斯基超空间 $\left(\mathbb{M}^{4|8}, D\right)$ 时，超共形 Killing 方程 (161) 等价于式 (10)，变换律 (163) 等价于式 (29)

Superconformal Action Principles

超共形作用量原理

To construct supergravity-matter systems, a locally superconformal action principle is required. Here we review three types of superconformal actions in $\mathcal{N} = 2$ supergravity that have played important roles in the literature.

要构建超引力-物质系统，需要局域超共形作用量原理。本文我们综述了 $\mathcal{N} = 2$ 超引力中三类已在文献中发挥重要作用的超共形作用量。

Full Superspace Action

全超空间作用量

The simplest locally superconformal action involves a full superspace integral:

最简单的局域超共形作用量包含一个全超空间积分:

$$S[\mathcal{L}] = \int \mathrm{d}^{4|8} z E \mathcal{L}, \ \mathrm{d}^{4|8} z \ := \ \mathrm{d}^4 x \ \mathrm{d}^4 \theta \mathrm{d}^4 \bar{\theta}, \ E \ := \ \mathrm{Ber}\left(E_M{}^A\right),$$

(164)

where $\mathcal{L}$ is a primary real dimensionless scalar Lagrangian,

其中 $\mathcal{L}$ 是一个主实无量纲标量拉格朗日量,

$$K^A \mathcal{L} = 0, \ \overline{\mathcal{L}} = \mathcal{L}, \ \mathbb{D}\mathcal{L} = 0. \tag{165}$$

As an example, we consider a superconformal higher-derivative σ -model with action [104, 109, 110]

我们举一个例子，考虑一个超共形高阶导数 σ 模型，其作用量为 [104, 109, 110]

$$S = \int \mathrm{d}^{4|8} z E \mathcal{K}\left(X^I, \bar{X}^{\bar{J}}\right), \ K^A X^I = 0, \ \bar{\nabla}_i^{\dot{\alpha}} X^I = 0, \ \mathbb{D} X^I = 0 \tag{166}$$

where $\mathcal{K}$ is the Kähler potential of a Kähler manifold. The action is locally superconformal. It is also invariant under Kähler transformations

其中 $\mathcal{K}$ 是凯勒流形的凯勒势。该作用量具有局域超共形不变性，同时也在凯勒变换下保持不变

$$\mathcal{K}\left(X, \bar{X}\right) \rightarrow \mathcal{K}\left(X, \bar{X}\right) + \Lambda(X) + \overline{\Lambda}\left(\bar{X}\right), \tag{167}$$

with $\Lambda(X)$ an arbitrary holomorphic function.

其中 $\Lambda(X)$ 是任意全纯函数。

Chiral Action

手征作用量

More general is the chiral action, which involves an integral over the chiral subspace

更一般的是手征作用量，它包含对手征子空间的积分

$$S_{\mathrm{c}}[\mathcal{L}_{\mathrm{c}}] = \int \mathrm{d}^4 x \ \mathrm{d}^4 \theta \mathcal{E} \mathcal{L}_{\mathrm{c}}. \tag{168}$$

Here $\mathcal{E}$ is a suitably chosen chiral measure, and $\mathcal{L}_c$ is a primary covariantly chiral Lagrangian of dimension +2,

此处 $\mathcal{E}$ 是恰当选取的手征测度，$\mathcal{L}_c$ 是维度为 +2 的本原协变手征拉格朗日量

$$K^A \mathcal{L}_c = 0, \ \bar{\nabla}^{\dot{\alpha}}_i \mathcal{L}_c = 0, \ \mathbb{D}\mathcal{L}_c = 2\mathcal{L}_c. \tag{169}$$

The precise definition of $\mathcal{E}$ in conformal superspace is somewhat technical [63]. In SU(2) superspace, $\mathcal{E}$ was obtained by making use of normal coordinates [58].

$\mathcal{E}$ 在共形超空间中的精确定义带有一定技术性 [63]。在 SU(2) 超空间中, $\mathcal{E}$ 可利用正规坐标得到 [58]。

A different definition of S_c exists, which is based on the use of a primary complex superfield Υ with the following superconformal properties (for some constant w):

S_c 存在另一种不同的定义，该定义基于使用满足如下超共形性质的本原复超场 Υ (对某个常数 w):

$$K^A \Upsilon = 0, \ \mathbb{D}\Upsilon = (w-2)\,\Upsilon, \ \mathbb{Y}\Upsilon = 2\,(2-w)\,\Upsilon, \tag{170}$$

such that $\bar{\nabla}^4 \Upsilon$ is nowhere vanishing, that is, $\left(\bar{\nabla}^4 \Upsilon\right)^{-1}$ exists. Specifically, the chiral action may be identified with the functional

使得 $\bar{\nabla}^4 \Upsilon$ 处处非零，即 $\left(\bar{\nabla}^4 \Upsilon\right)^{-1}$ 存在。具体来说，手征作用量可表示为如下泛函

$$S_c\,[\mathcal{L}_c] = \int \mathrm{d}^{4|8} z E \frac{\Upsilon}{\bar{\nabla}^4 \Upsilon} \mathcal{L}_c, \tag{171}$$

which possesses the two fundamental properties: (i) it is locally superconformal under the conditions (169), and (ii) it is independent of Υ ,

它具有两个基本性质:(i) 在条件 (169) 下是局部超共形的，(ii) 它与 Υ 无关,

$$\delta_\Upsilon \int \mathrm{d}^{4|8} z E \frac{\Upsilon}{\bar{\nabla}^4 \Upsilon} \mathcal{L}_c = 0, \tag{172}$$

for an arbitrary variation $\delta\Upsilon$. Using the representation (171) for the chiral action (168), it holds that

对任意变分 $\delta\Upsilon$ 都成立。利用表示式 (171) 改写手征作用量 (168)，可得

$$\int \mathrm{d}^{4|8} z E \mathcal{L} = \int \mathrm{d}^4 x \ \mathrm{d}^4\theta \mathcal{E} \mathcal{L}_c, \ \mathcal{L}_c = \bar{\nabla}^4 \mathcal{L}. \tag{173}$$

There is an alternative definition of the chiral action that follows from the superform approach to the construction of supersymmetric invariants [111-113]. It is based on the use of the following super 4-form [114]:

手征作用量还有另一种定义，源自构造超对称不变量的超形方法 [111-113]，该定义基于使用如下4-超形式 [114]:

$$\Xi_4 = -4E^j_\beta \wedge E^{\dot\beta}_j \wedge E^i_{\dot\alpha} \wedge E^{\dot\alpha}_i \mathcal{L}_c - 2E^j_\beta \wedge E^{\dot\beta}_j \wedge E^i_{\dot\alpha} \wedge E^a (\tilde\sigma_a)^{\dot\alpha\alpha} \nabla_{\alpha i}\mathcal{L}_c$$

$$-\frac{i}{2}E^j_{\dot\beta} \wedge E^i_{\dot\alpha} \wedge E^b \wedge E^a (\tilde\sigma_{ab})^{\dot\alpha\dot\beta} \nabla^{ij}\mathcal{L}_c$$

$$-\frac{i}{4}E^i_{\dot\alpha} \wedge E^{\dot\alpha}_i \wedge E^b \wedge E^a \left((\sigma_{ab})_{\alpha\beta}\nabla^{\alpha\beta} - 8(\tilde\sigma_{ab})_{\dot\alpha\dot\beta}\bar W^{\dot\alpha\dot\beta}\right)\mathcal{L}_c$$

$$-\frac{i}{36}\varepsilon_{abcd}E^i_{\dot\alpha} \wedge E^c \wedge E^b \wedge E^a \left(\left(\tilde\sigma^d\right)^{\dot\alpha\alpha}\nabla^j_\alpha\nabla_{ij} - 6\left(\tilde\sigma^d\right)^{\dot\beta\alpha}\bar W_{\dot\alpha\dot\beta}\nabla_{\alpha i}\right)\mathcal{L}_c$$

$$+\frac{1}{24}\varepsilon_{abcd}E^d \wedge E^c \wedge E^b \wedge E^a \left(\nabla^4 + \bar W^{\dot\alpha\dot\beta}\bar W_{\dot\alpha\dot\beta}\right)\mathcal{L}_c \tag{174}$$

This superform is closed,

该超形式是闭形式,

$$\mathrm{d}\,\Xi_4 = 0\,. \tag{175}$$

It proves to be primary (The superform may be degauged to SU (2) superspace. Then condition (176) is equivalent to the super-Weyl invariance of Ξ_4 .)

可以证明它是本原的 (该超形可退规范至 SU (2) 超空间。此时条件 (176) 等价于 Ξ_4 的超外尔不变性。)

$$K^B\Xi_4 = 0. \tag{176}$$

The chiral action (168) can be recast as an integral of Ξ_4 over a spacetime $\mathcal{M}^4$,

手征作用量 (168) 可以改写为 Ξ_4 在时空子流形 $\mathcal{M}^4$ 上的积分,

$$S_c\,[\mathcal{L}_c] = \int_{\mathcal{M}^4}\Xi_4, \tag{177a}$$

where $\mathcal{M}^4$ is the bosonic body of the curved superspace $\mathcal{M}^{4|4}$ obtained by switching off the Grassmann variables. It turns out that (177a) leads to the following representation [63] (see also [64]):

其中 $\mathcal{M}^4$ 是关闭格拉斯曼变量后得到的弯曲超空间 $\mathcal{M}^{4|4}$ 的玻恩主体。可以证明 (177a) 导出如下表示 [63](也见 [64]):

$$S_c = \int \mathrm{d}^4x e\Big(\nabla^4 + \bar W^{\dot\alpha\dot\beta}\bar W_{\dot\alpha\dot\beta} - \frac{i}{12}\overline{\psi}_{d\dot\delta}\left(\left(\tilde\sigma^d\right)^{\dot\delta\alpha}\nabla^q_\alpha\nabla_{lq} - 6\left(\sigma^d\right)_{\alpha\dot\alpha}\bar W^{\dot\alpha\dot\delta}\nabla^\alpha_l\right)$$

$$+\frac{1}{4}\overline{\psi}_{c\dot{\gamma}}{}^{k}\overline{\psi}_{d\dot{\delta}}{}^{l}\Big((\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}}\nabla_{kl} - \frac{1}{2}\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon_{kl}(\sigma^{cd})_{\beta\gamma}\nabla^{\beta\gamma} - 4\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon_{kl}(\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}}\bar{W}^{\dot{\alpha}\dot{\beta}}\Big)$$

$$-\frac{1}{4}\varepsilon^{abcd}(\tilde{\sigma}_a)^{\dot{\beta}\alpha}\overline{\psi}_{b\dot{\beta}}{}^{j}\overline{\psi}_{c\dot{\gamma}}{}^{k}\overline{\psi}_{dk}{}^{\dot{\gamma}}\nabla_{\alpha j} - \frac{\mathrm{i}}{4}\varepsilon^{abcd}\overline{\psi}_{a\dot{\alpha}}{}^{i}\overline{\psi}_{bi}{}^{\dot{\alpha}}\overline{\psi}_{c\dot{\beta}}{}^{j}\overline{\psi}_{dj}{}^{\dot{\beta}})\mathcal{L}_{\mathrm{c}}{}^{\dot{\beta}}\Big|_{\theta=0} \tag{177b}$$

where $e := \det(e_m{}^a)$. This result agrees with the action of a chiral multiplet coupled to conformal supergravity [115].

其中 $e := \det(e_m{}^a)$。该结果与耦合共形超引力的手征多重态作用量一致 [115]。

Projective Action

投影作用量

Consider a Lagrangian $\mathcal{L}^{(2)}$ that is a real weight-2 projective multiplet. Associated with $\mathcal{L}^{(2)}$ is the action

考虑拉格朗日量 $\mathcal{L}^{(2)}$，它是一个实 2 权投影多重态。与 $\mathcal{L}^{(2)}$ 对应的作用量为

$$S\left[\mathcal{L}^{(2)}\right] = \frac{1}{2\pi}\oint_{\gamma}(v,\ \mathrm{d}v)\int \mathrm{d}^{4|8}zE\frac{\Upsilon^{(n)}}{\nabla^{(4)}\Upsilon^{(n)}}\mathcal{L}^{(2)},\ (v,\ \mathrm{d}v) := v^i\,\mathrm{d}v_i, \tag{178}$$

where $\Upsilon^{(n)}(z,v)$ is a primary weight- n isotwistor superfield and the operator $\nabla^{(4)}$ is defined in (120). This action proves to have the following fundamental properties: (i) it is locally superconformal, and (ii) it is independent of $\Upsilon^{(n)}$,

其中 $\Upsilon^{(n)}(z,v)$ 是本原权- n 同扭超场，算符 $\nabla^{(4)}$ 定义于式 (120)。该作用量具备以下基本性质:(i) 它是局部超共形的，(ii) 它与 $\Upsilon^{(n)}$ 无关，

$$\delta_{\Upsilon^{(n)}}\oint_{\gamma}(v,\ \mathrm{d}v)\int \mathrm{d}^{4|8}zE\frac{\Upsilon^{(n)}}{\nabla^{(4)}\Upsilon^{(n)}}\mathcal{L}^{(2)} = 0. \tag{179}$$

In the $n = 0$ case we can specialize $\Upsilon^{(0)}$ to be $W_0\bar{W}_0$, where W_0 is the chiral field strength of a vector multiplet, see section "Vector Multiplet," such that the descendant

在 $n = 0$ 的情形下，我们可以将 $\Upsilon^{(0)}$ 特殊化为 $W_0\bar{W}_0$，其中 W_0 是向量多重态的手征场强，参见“向量多重态”一节，此时降算符

$$\textstyle\sum_0{}^{ij} := \frac{1}{4}\nabla^{ij}W_0 = \frac{1}{4}\overline{\nabla}^{ij}\bar{W}_0 \tag{180}$$

is nowhere vanishing, that is, $\left(\sum_0{}^{ij}\sum_{0ij}\right)^{-1}$ exists. Then (178) turns into [59]

处处非零，即 $\left(\sum_0{}^{ij}\sum_{0ij}\right)^{-1}$ 存在。那么式 (178) 就变为 [59]

$$S\left[\mathcal{L}^{(2)}\right] = \frac{1}{2\pi}\oint_{\gamma}(v,\ \mathrm{d}v)\int \mathrm{d}^{4|8}zE\frac{W_0\bar{W}_0}{\left(\sum_0{}^{(2)}\right)^2}\mathcal{L}^{(2)}. \tag{181}$$

An important remark is in order. In the case of Minkowski superspace, it may be seen that the definition of the projective action (178) is equivalent to (54).

有一个重要说明需要指出。在闵可夫斯基超空间的情形下，可以看出投影作用量的定义式 (178) 与式 (54) 等价。

There is a remarkable relationship between the projective and the chiral actions [57,58] derived originally in SU(2) superspace. It makes use of the vector multiplet introduced above. For every chiral Lagrangian $\mathcal{L}_{\mathrm{c}}$ with the properties (169), the chiral action

投影作用量和手征作用量之间存在一个引人注目的关系 [57,58]，该关系最初在 SU(2) 超空间中推导得到。它利用了上文引入的向量多重态。对任意满足性质 (169) 的手征拉格朗日量 $\mathcal{L}_{\mathrm{c}}$，手征作用量

$$S_{\text{chiral}}=\int \mathrm{d}^4 x\, \mathrm{d}^4 \theta \mathcal{E} \mathcal{L}_{\mathrm{c}}+\text{ c.c.} \tag{182}$$

can be represented as a projective action

可以表示为投影作用量

$$\begin{gathered}S_{\text{chiral}}=\frac{1}{2\pi}\oint_\gamma (v,\ \mathrm{d}v)\int \mathrm{d}^{4|8}zE\frac{W_0\bar{W}_0}{\left(\Sigma_0^{(2)}\right)^2}\mathcal{L}_{\mathrm{c}}^{(2)},\\ \mathcal{L}_{\mathrm{c}}^{(2)}=-\frac{1}{4}V\left(\nabla^{(2)}\frac{\mathcal{L}_{\mathrm{c}}}{W_0}+\bar{\nabla}^{(2)}\frac{\bar{\mathcal{L}}_{\mathrm{c}}}{\bar{W}_0}\right)\equiv V\mathfrak{G}^{(2)}.\end{gathered} \tag{183}$$

Here $V(z,v)$ is a tropical prepotential for the vector multiplet with the chiral field strength W_0 ; see the next section.

此处 $V(z,v)$ 是带手征场强 W_0 的向量多重态的热带预势；参见下一节。

On the other hand, the projective action (181) can be rewritten as a special chiral action [58]

另一方面，投影作用量 (181) 可以改写为特殊手征作用量 [58]

$$S\left[\mathcal{L}^{(2)}\right]=\int \mathrm{d}^4x\, \mathrm{d}^4\theta\mathcal{E}W_0\mathbb{W},\ \mathbb{W}=\frac{1}{8\pi}\oint(v,\ \mathrm{d}v)\,\bar{\nabla}^{(-2)}\left(\frac{\mathcal{L}^{(2)}}{\Sigma_0\,(2)}\right), \tag{184}$$

with the operator $\nabla^{(-2)}$ being defined in (192). The composite superfield $\mathbb{V}$ can be interpreted as a tropical prepotential for the vector multiplet described by the reduced chiral superfield $\mathbb{W}$.

其中算符 $\nabla^{(-2)}$ 定义于式 (192)。复合超场 $\mathbb{V}$ 可以诠释为由约化手征超场 $\mathbb{W}$ 描述的向量多重态的热带预势。

An important example of a dynamical system described by the projective action is provided by the off-shell sigma model (58), in which $\Upsilon^{(1)}$ and $\breve{\Upsilon}^{(1)}$ are now covariant arctic and antarctic multiplets, respectively. This most general locally superconformal sigma model was studied in detail in [66], where its component reduction was worked out.

投影作用量描述的动力学系统的一个重要例子是脱壳 sigma 模型 (58)，其中 $\Upsilon^{(1)}$ 和 $\breve{\Upsilon}^{(1)}$ 分别是协变北极多重态和南极多重态。这个最一般的局部超共形 sigma 模型已在文献 [66] 中得到详细研究，其分量约化也在该文献中完成推导。

Vector and Tensor Multiplets

向量多重态与张量多重态

Of special importance in $\mathcal{N} = 2$ supersymmetry are vector and tensor multiplets. Here we review their fundamental properties in the framework of conformal superspace.

在 $\mathcal{N} = 2$ 超对称中，向量多重态与张量多重态具有特殊重要性。本文我们在共形超空间的框架下回顾它们的基本性质。

Vector Multiplet

向量多重态

In rigid supersymmetry, the off-shell $\mathcal{N} = 2$ vector multiplet was formulated by Grimm, Sohnius, and Wess [38]. In conformal superspace, it can be described by a field strength W , which has the superconformal properties

在刚性超对称中，离壳 $\mathcal{N} = 2$ 向量多重态由 Grimm、Sohnius 和 Wess 提出 [38]。在共形超空间中，它可由场强 W 描述，该场强具有超共形性质

$$K^A W = 0,\ \mathbb{D}W = W,\ \bar{\nabla}^{\dot{\alpha}}_i W = 0 \tag{185a}$$

and satisfies the Bianchi identity

并满足 Bianchi 恒等式

$$\sum^{ij} := \frac{1}{4}\nabla^{ij} W = \frac{1}{4}\bar{\nabla}^{ij} \bar{W}. \tag{185b}$$

Covariantly chiral scalars satisfying the reality condition (185b) are called reduced chiral. The constraint (185b) uniquely determines the dimension of W .

满足实条件 (185b) 的协变手征标量被称为约化手征标量。约束条件 (185b) 唯一确定了 W 的量纲。

There are several ways to realize W as a gauge-invariant field strength. One possibility is to introduce a curved superspace extension of Mezincescu's prepotential [116] (see also [117]), $V_{ij} = V_{ji}$, which is a primary unconstrained real SU(2) triplet of dimension -2 . The expression for W in terms of V_{ij} [118] is

存在多种将 W 实现为规范不变场强的方法。一种可能是引入 Mezincescu 预势的弯曲超空间推广 [116](另见 [117])，即 $V_{ij} = V_{ji}$，它是量纲为-2 的本原无约束实 SU(2) 三重态。用 V_{ij} [118] 表示 W 的表达式为

$$W = \frac{1}{4}\bar{\nabla}^4 \nabla^{ij} V_{ij} \tag{186}$$

where the chiral operator $\bar{\nabla}^4$ is defined in (109). It may be shown that V_{ij} is defined only up to gauge transformations of the form

其中手征算符 $\bar{\nabla}^4$ 定义于 (109)。可以证明，V_{ij} 仅在如下形式的规范变换下良定义

$$\delta V^{ij} = \nabla^{\alpha}{}_k \Lambda_\alpha{}^{kij} + \bar{\nabla}_{\dot{\alpha}k} \bar{\Lambda}^{\dot{\alpha}kij}, \ \Lambda_\alpha{}^{kij} = \Lambda_\alpha{}^{(kij)}, \ \bar{\Lambda}^{\dot{\alpha}}{}_{kij} := \overline{\Lambda_\alpha{}^{kij}}, \tag{187}$$

with the primary gauge parameter $\Lambda_\alpha{}^{kij}$ being completely arbitrary modulo the algebraic condition given. The superconformal properties of $\Lambda_\alpha{}^{kij}$ are determined by those of V^{ij} .

本原规范参数 $\Lambda_\alpha{}^{kij}$ 在给定代数条件下完全任意。$\Lambda_\alpha{}^{kij}$ 的超共形性质由 V^{ij} 的性质决定。

Let us show how Mezincescu's prepotential for the vector multiplet can be introduced within standard superspace. For this a simple generalization of the rigid supersymmetric analysis in [117] can be used. One begins with the first-order action

下面我们说明如何在标准超空间框架下引入向量多重态的 Mezincescu 预势。为此可以使用文献 [117] 中刚性超对称分析的一个简单推广。我们从一阶作用量出发

$$S = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^4\theta \varepsilon \mathcal{W}\mathcal{W} + \text{ c.c. } - \frac{\mathrm{i}}{8}\int \mathrm{d}^{4|8}z E\Big(\mathcal{W}\nabla^{ij}V_{ij} - \bar{\mathcal{W}}\bar{\nabla}^{ij}V_{ij}\Big),$$

(188)

where $\mathcal{W}$ is a covariantly chiral superfield and $V^{ij} = V^{ji}$ is an unconstrained real SU(2) triplet acting as a Lagrange multiplier. Varying (188) with respect to V_{ij} gives $\mathcal{W} = W$, where W obeys the Bianchi identity (185b). As a result, the second term in (188) drops out and we end up with the $\mathcal{N} = 2$ super-Maxwell action

其中 $\mathcal{W}$ 是协变手征超场，$V^{ij} = V^{ji}$ 是作为拉格朗日乘子的无约束实 SU(2) 三重态。将 (188) 对 V_{ij} 变分得到 $\mathcal{W} = W$，其中 W 满足 Bianchi 恒等式 (185b)。最终 (188) 中的第二项消失，我们得到 $\mathcal{N} = 2$ 超麦克斯韦作用量

$$S = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^4\theta \mathcal{E} W W + \text{ c.c.} \tag{189}$$

On the other hand, because the action (188) is quadratic in $\mathcal{W}$, we may easily integrate $\mathcal{W}$ out using its equation of motion

另一方面，由于作用量 (188) 对 $\mathcal{W}$ 是二次的，我们可以利用其运动方程轻松积出 $\mathcal{W}$

$$
\mathcal{W} = \mathrm{i}W_{\mathrm{D}}, \; W_{\mathrm{D}} \; := \; \frac{1}{4}\bar{\nabla}^{4}\nabla^{ij}V_{ij}. \tag{190}
$$

This leads to the dual action

这给出对偶作用量

$$
S = \frac{1}{4}\int \mathrm{d}^4x \; \mathrm{d}^4\theta \mathcal{E} W_{\mathrm{D}} W_{\mathrm{D}} + \text{ c.c.} \tag{191}
$$

The dual field strength W_{D} must be both reduced chiral and given by (190).

对偶场强 W_{D} 必须同时是约化手征的，且满足 (190)。

Within the curved projective-superspace approach of [56, 57, 59] , the constraints on W can be solved in terms of a covariant real weight- 0 tropical prepotential $V\left(v^i\right)$, $\breve{V} = V$. The solution [58] is

在 [56, 57, 59] 的弯曲投影超空间方法中，W 上的约束可以用协变实零权热带预势 $V\left(v^i\right)$、$\breve{V} = V$ 求解。解 [58] 为

$$
W = \frac{1}{8\pi}\oint_{\gamma} (v, \; \mathrm{d}v)\,\bar{\nabla}^{(-2)} V(v), \; \bar{\nabla}^{(-2)} \; := \; \frac{1}{(v,u)^2} u_i u_j \bar{\nabla}^{ij}. \tag{192}
$$

where γ is an appropriately chosen contour. We recall that $v^i \in \mathbb{C}^2 \setminus \{0\}$ denotes the homogeneous coordinates for $\mathbb{C}P^1$. The right-hand side of the expression for W involves a constant isotwistor u_i , which is chosen to obey the constraint $(v, u) \; := \; v^i u_i \neq 0$, but otherwise is completely arbitrary. Using the analyticity constraints (113) obeyed by V , one can check that W is invariant under arbitrary projective transformations (56). The field strength (192) proves to be invariant under gauge transformations

其中 γ 是适当选取的围道。我们回顾，$v^i \in \mathbb{C}^2 \setminus \{0\}$ 表示 $\mathbb{C}P^1$ 的齐次坐标。W 表达式的右侧包含一个常数等扭量 u_i ，它被要求满足约束 $(v, u) \; := \; v^i u_i \neq 0$ ，除此之外完全任意。利用 V 满足的解析性约束 (113)，可以验证 W 在任意投影变换 (56) 下不变。场强 (192) 在如下规范变换下保持不变

$$
V \to V + \lambda + \breve{\lambda} \tag{193}
$$

where the gauge parameter $\lambda(v)$ is a covariant weight-0 arctic multiplet.

其中规范参数 $\lambda(v)$ 是协变零权北极多重态。

It is worth discussing how the Mezincescu prepotential V_{ij} emerges within projective superspace; see [118] for more details. One begins with the expression for W in terms of $V(v)$, Eq. (192). In accordance with Eq. (118), the analyticity conditions on V may be solved in terms of an unconstrained isotwistor superfield $U^{(-4)}$, which is real under smile-conjugation

值得讨论梅津采夫预势 V_{ij} 是如何在投影超空间中出现的；更多细节见文献 [118]。我们从用 $V(v)$ 表示 W 的表达式，即式 (192) 出发。根据式 (118)，V 的解析性条件可以用无约束同扭超场 $U^{(-4)}$ 求解，该超场在微笑共轭下是实的

$$V(v) = \frac{1}{16}\bar{\nabla}^{(2)}\nabla^{(2)}U^{(-4)}(v) = \frac{1}{16}\nabla^{(2)}\bar{\nabla}^{(2)}U^{(-4)}(v)\,. \tag{194}$$

Using this construction, one may rewrite W as

利用该构造，我们可以将 W 重写为

$$W = \frac{1}{128\pi}\oint_\gamma (v,\ \mathrm{d}v)\,\bar{\nabla}^{(-2)}\bar{\nabla}^{(2)}\nabla^{(2)}U^{(-4)} = \frac{1}{8\pi}\bar{\nabla}^4\oint_\gamma (v,\ \mathrm{d}v)\,\nabla^{(2)}U^{(-4)}$$

(195)

where the chiral operator $\bar{\nabla}^4$ is defined in (109). This may be rewritten as

其中手征算符 $\bar{\nabla}^4$ 定义在式 (109) 中，上式可改写为

$$W = \frac{1}{8\pi}\bar{\nabla}^4\nabla^{ij}\oint_\gamma (v,\ \mathrm{d}v)\,v_i v_j U^{(-4)}(v) = \frac{1}{4}\bar{\nabla}^4\nabla^{ij}V_{ij}, \tag{196}$$

where we have defined the Mezincescu prepotential

其中我们定义了梅津采夫预势

$$V_{ij} = \frac{1}{2\pi}\oint_\gamma (v,\ \mathrm{d}v)\,v_i v_j U^{(-4)}(v)\,. \tag{197}$$

Given a system of n Abelian vector multiplets with chiral field strengths W_I, let $\mathcal{F}(W)$ be a holomorphic function of degree +2,

给定 n 个阿贝尔向量多重态构成的系统，其手征场强为 W_I，设 $\mathcal{F}(W)$ 是次数为 +2 的全纯函数，

$$W_I\frac{\partial}{\partial W_I}\mathcal{F}(W) = 2\mathcal{F}(W)\,. \tag{198}$$

Then the following action

则如下作用量

$$S = \int \mathrm{d}^4x\ \mathrm{d}^4\theta\,\mathcal{E}\mathcal{F}(W) + \text{ c.c.} \tag{199}$$

is locally superconformal. The component reduction of this model was described by Butter and Novak [64], and their results agree with [17]. The model has led to the notion of special Kähler geometry [119]; see [18] for a review. A rigid supersymmetric limit of (199) corresponds to rigid special Kähler geometry [120, 121].

是局部超共形的。该模型的分量约化由巴特和诺瓦克 [64] 给出，他们的结果与文献 [17] 一致。该模型引出了特殊凯勒几何的概念 [119]；综述见文献 [18]。式 (199) 的刚性超对称极限对应刚性特殊凯勒几何 [120, 121]。

Tensor Multiplet

张量多重态

In rigid supersymmetry, the massless $\mathcal{N}=2$ tensor multiplet was introduced by Wess [81]. It was rediscovered by de Wit and van Holten [5]. The tensor multiplet can be described in conformal superspace by its gauge-invariant field strength G^{ij} , which is a real $\mathcal{O}(2)$ multiplet. It obeys the constraints

在刚性超对称中，无质量 $\mathcal{N}=2$ 张量多重态由韦斯引入 [81]，后由德维特与范霍尔顿重新发现 [5]。张量多重态可在共形超空间中通过其规范不变场强 G^{ij} 描述，G^{ij} 是实 $\mathcal{O}(2)$ 多重态，满足如下约束

$$\nabla_\alpha^{(i}G^{jk)} = \bar{\nabla}_{\dot\alpha}^{(i}G^{jk)} = 0, \tag{200}$$

which generalize those given in [20, 24, 82] . These constraints are solved in terms of a chiral prepotential Ψ with the superconformal properties

这些约束是对 [20, 24, 82] 中已有约束的推广，可通过具有超共形性质的手征预备势 Ψ 求解

$$K^A\Psi = 0,\ \mathbb{D}\Psi = \Psi,\ \bar{\nabla}_i^{\dot\alpha}\Psi = 0. \tag{201}$$

The solution to the tensor multiplet constraints was given in [25, 117, 122, 123] . In conformal superspace the solution is

张量多重态约束的解最早给出于 [25, 117, 122, 123] ，共形超空间中的解为

$$G^{ij} = \frac{1}{4}\nabla^{ij}\Psi + \frac{1}{4}\bar{\nabla}^{ij}\bar{\Psi}. \tag{202}$$

The chiral prepotential is invariant under gauge transformations

手征预备势在规范变换下不变

$$\Psi \to \Psi + \mathrm{i}\Lambda, \tag{203}$$

where the gauge parameter Λ is a reduced chiral superfield with the properties (185).

其中规范参数 Λ 是约化手征超场，具有性质 (185)。

Consider a system of $(n+1)$ tensor multiplets, $n>0$, and let $G_I^{(2)}$ be their gauge-invariant field strengths, $I=0,1,\ldots,n$. Its dynamics can be described by a Lagrangian of the form

考虑由 $(n+1)$ 个张量多重态组成的系统 $n>0$ ，记 $G_I^{(2)}$ 为它们的规范不变场强 $I=0,1,\ldots,n$ ，其动力学可由如下形式的拉格朗日量描述

$$\mathcal{L}^{(2)} = \mathcal{L}\left(G_I^{(2)}\right),\ G_I^{(2)}\frac{\partial}{\partial G_I^{(2)}}\mathcal{L} = \mathcal{L} \tag{204}$$

where $\mathcal{L}$ is a real homogeneous function of degree $+1$. Of special significance is the special choice of $\mathcal{L}$ defined by the Lagrangian

其中 $\mathcal{L}$ 是一次齐次实函数。由拉格朗日量定义的下述特殊选择的 $\mathcal{L}$ 具有特殊意义:

$$\mathcal{L}^{(2)}=\frac{1}{\mathrm{i}G_0^{(2)}}\left(\mathcal{F}\left(G_I^{(2)}\right)-\overline{\mathcal{F}}\left(G_I^{(2)}\right)\right). \tag{205}$$

Here $\mathcal{F}\left(z^I\right)$ is a holomorphic homogeneous function of second degree, $\mathcal{F}\left(\mathfrak{c}z^I\right)=\ \mathfrak{c}^2\mathcal{F}\left(z^I\right)$. This model provides a manifestly supersymmetric description of the c-map [101, 102]. The rigid c-map is described by the model (81).

此处 $\mathcal{F}\left(z^I\right)$ 是二次齐次全纯函数 $\mathcal{F}\left(\mathfrak{c}z^I\right)=\ \mathfrak{c}^2\mathcal{F}\left(z^I\right)$。该模型给出了 c 映射的明显超对称描述 [101, 102]，刚性 c 映射由模型 (81) 描述。

For a single tensor multiplet there is only one superconformal model, which is described by the Lagrangian

单个张量多重态仅存在一种超共形模型，由如下拉格朗日量描述

$$\mathcal{L}_{\mathrm{IT}}^{(2)}=-G^{(2)}\ln\frac{G^{(2)}}{\mathrm{i}\gamma^{(1)}\breve{\gamma}^{(1)}}, \tag{206}$$

with $\Upsilon^{(1)}$ a weight-one arctic multiplet (both $\Upsilon^{(1)}$ and its smile-conjugate $\breve{\gamma}(1)$ are pure gauge degrees of freedom). It describes an improved tensor multiplet. Historically, the improved tensor multiplet was independently constructed in the following works (submitted to the journal Nuclear Physics within a 1 day time difference): (i) Ref. [37] provided its construction in terms of $\mathcal{N}=1$ superfields in the rigid supersymmetric case, and (ii) and Ref. [15] proposed this multiplet within the $\mathcal{N}=2$ superconformal tensor calculus. The rigid supersymmetric version of (206) was proposed in the first projective-superspace paper [37].

其中 $\Upsilon^{(1)}$ 是权重为 1 的北极多重态 ($\Upsilon^{(1)}$ 及其微笑共轭 $\breve{\gamma}(1)$ 都是纯规范自由度)，它描述改进张量多重态。历史上，改进张量多重态由以下两项工作独立构造 (二者投稿至《核物理》期刊的时间差不超过 1 天):(i) 文献 [37] 在刚性超对称情形下用 $\mathcal{N}=1$ 超场给出了构造，(ii) 文献 [15] 在 $\mathcal{N}=2$ 超共形张量演算框架下提出了该多重态。(206) 的刚性超对称版本最早在投影超空间的开篇论文 [37] 中提出。

The improved tensor multiplet can be coupled to weight-0 polar hypermultiplets. The corresponding locally superconformal σ -model [57] is

改进张量多重态可以与权重为 0 的极超多重态耦合，对应的局域超共形 σ 模型 [57] 为

$$\mathcal{L}^{(2)}=-G^{(2)}\ln\frac{G^{(2)}}{\mathrm{i}\breve{\Upsilon}^{(1)}\Upsilon^{(1)}}+G^{(2)}K\left(\Upsilon,\breve{\Upsilon}\right), \tag{207}$$

where the Kähler potential is the same as in (66). The rigid supersymmetric limit of this σ -model was studied in [125]. The above σ -model has a dual formulation in terms of polar hypermultiplets:

其中凯勒势与 (66) 中相同。该 σ 模型的刚性超对称极限已在文献 [125] 中研究。上述 σ 模型存在极超多重态语言下的对偶表述:

$$\mathcal{L}^{(2)} = \mathrm{i}\breve{\gamma}^{(1)}\gamma^{(1)}\mathrm{e}^{K(\Upsilon,\breve{\Upsilon})}. \tag{208}$$

The locally $\mathcal{N} = 2$ superconformal σ -models (207) and (208) have a striking resemblance to their $\mathcal{N} = 1$ counterparts; see, e.g.,[126].

局域 $\mathcal{N} = 2$ 超共形 σ 模型 (207) 和 (208) 与它们的 $\mathcal{N} = 1$ 对应模型惊人地相似, 参见例如文献 [126]。

Linear Multiplet Action

线性多重态作用量

The linear multiplet action is a BV-type superconformal invariant based on the Lagrangian

线性多重态作用量是基于拉格朗日量的 BV 型超共形不变量

$$\mathcal{L}^{(2)} = VG^{(2)}. \tag{209}$$

There are three equivalent representations for the linear multiplet action:

线性多重态作用量存在三种等价表示:

$$S\left[VG^{(2)}\right] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\mathcal{E}W\Psi + \text{ c.c. } = \int \mathrm{d}^{4|8}zEV_{ij}G^{ij}. \tag{210}$$

The action is invariant under the gauge transformations for the vector and tensor multiplets. The invariance under (193) follows from the identity

该作用量在矢量多重态和张量多重态的规范变换下不变。(193) 式下的不变性可由下述恒等式推出

$$S\left[\left(\lambda + \breve{\lambda}\right)G^{(2)}\right] = 0, \tag{211}$$

where λ is an arctic multiplet. The invariance under (203) follows from the Bianchi identity (185b).

其中 λ 是一个北极多重态。(203) 式下的不变性可由比安基恒等式 (185b) 推出。

We have seen that every chiral action can be represented as a projective action, Eq. (183). On the other hand every projective action can be recast as a chiral action, Eq. (184). These results show that the linear multiplet action (210) is universal.

我们已经知道，每个手征作用量都可以表示为投影作用量，即式 (183)。另一方面，每个投影作用量都可以改写为手征作用量，即式 (184)。这些结果表明，线性多重态作用量 (210) 是通用的。

Composite Reduced Chiral Superfields

复合约化手征超场

The above discussion has an important implication. Given a composite real weight-0 tropical multiplet $\mathbb{V}$, the following descendant

上述讨论有一个重要推论: 给定一个实 0 权复合热带多重态 $\mathbb{V}$，下述导出场

$$\mathbb{W} = \frac{1}{8\pi}\oint_{\gamma}(v,\ \mathrm{d}v)\,\overline{\nabla}^{(-2)}\mathbb{V}(v) \tag{212}$$

is a primary reduced chiral superfield, with γ being an appropriately chosen contour. This observation has been used in [118] to derive a number of composite reduced chiral superfields.

是一个基本约化手征超场，其中 γ 为适当选取的围道。文献 [118] 利用这一结论构造了多个复合约化手征超场。

Our first example is

我们的第一个例子是

$$\mathbb{V} = \ln\frac{G^{(2)}}{\mathrm{i}\Upsilon^{(1)}\breve{\Upsilon}^{(1)}}. \tag{213}$$

It can be seen that the arctic multiplet $\Upsilon^{(1)}$ and its conjugate $\breve{\Upsilon}^{(1)}$ do not contribute to the contour integral, and so they will be ignored below. Evaluating the contour integral in (212) gives

可以证明，北极多重态 $\Upsilon^{(1)}$ 及其共轭 $\breve{\Upsilon}^{(1)}$ 对围道积分没有贡献，因此下文将忽略它们。计算 (212) 式的围道积分可得

$$\mathbb{W} \;:=\; -\frac{G}{8}\overline{\nabla}_{ij}\left(\frac{G^{ij}}{G^2}\right),\ G \;:=\; \sqrt{\frac{1}{2}G^{ij}G_{ij}}; \tag{214}$$

see [118] for the technical details. This composite multiplet was discovered originally (in a different but equivalent form) in [15] using the superconformal tensor calculus. It was later reconstructed in curved superspace by Müller [123] with the aid of the results in [15] and [90]. Its contour origin was explored in the globally supersymmetric case by Siegel [90].

技术细节参见文献 [118]。该复合多重态最初由文献 [15] 利用超共形张量微积分 (以一种不同但等价的形式) 发现，后来穆勒 (Müller) 借助文献 [15] 和 [90] 的结果在弯曲超空间中重构了该多重态，西格尔 (Siegel) 在整体超对称情形下研究了它的围道积分起源。

Our second example is

我们的第二个例子是

$$\mathbb{V} = \frac{H^{(2n)}}{\left(G^{(2)}\right)^n}, \tag{215}$$

where $H^{(2n)}$ is a real $\mathcal{O}(2n)$ multiplet; see Eqs. (115) and (116). Evaluating the contour integral in (212) gives

其中 $H^{(2n)}$ 是一个实 $\mathcal{O}(2n)$ 多重态，参见式 (115) 和 (116)。计算 (212) 式的围道积分可得

$$\mathbb{W}_n = -\frac{(2n)!}{2^{2n+2}(n+1)!(n-1)!} G\bar{\nabla}_{ij}\mathcal{R}_n^{ij}, \tag{216}$$

where

其中

$$\mathcal{R}_n^{ij} = \left(\delta_k^i\delta_l^j - \frac{1}{2G^2}G^{ij}G_{kl}\right)H^{kli_1\cdots i_{2n-2}}G_{i_1i_2}\cdots G_{i_{2n-3}i_{2n-2}}G^{-2n}. \tag{217}$$

The expression for $\mathbb{W}_n$ has an overall structure similar to (214), except the argument $\mathcal{R}_n^{ij}$ of the derivative is much more complicated.

$\mathbb{W}_n$ 的整体结构与式 (214) 类似，只是导数的自变量 $\mathcal{R}_n^{ij}$ 复杂得多。

Off-Shell Formulations for Supergravity

超引力的离壳 formulation

Within the conformal approach to locally supersymmetric theories [9], Poincaré and AdS supergravity may be realized as conformal supergravity coupled to a compensating multiplet. Two compensating massless multiplets are typically required in the case of $\mathcal{N}=2$ supergravity; see [15, 18] for comprehensive discussions. In this section we describe two off-shell formulations for $\mathcal{N}=2$ supergravity.

在局域超对称理论的共形方法 [9] 中，庞加莱与反德西特超引力可实现为耦合补偿多重态的共形超引力。对于 $\mathcal{N}=2$ 超引力，通常需要两个无质量补偿多重态；全面讨论见 [15, 18] 。本节我们介绍 $\mathcal{N}=2$ 超引力的两种离壳 formulation。

Supergravity with Vector and Tensor Multiplet Compensators

带有向量和张量多重态补偿子的超引力

The minimal formulation for $\mathcal{N}=2$ supergravity with vector and tensor compensators [15] admits a simple superspace description. Using the techniques developed above, the gauge-invariant supergravity action can be written as

$\mathcal{N}=2$ 带向量和张量补偿子的极小超引力表述 [15] 可以用简单的超空间描述。利用上文开发的技术，规范不变的超引力作用量可写为

$$
\begin{aligned}
S_{\text{SUGRA}} &= \frac{1}{\kappa^2}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\mathcal{E}\left\{\Psi\mathbb{W} - \frac{1}{4}W^2 + \xi\Psi W\right\} + \text{ c.c.} \\
&= \frac{1}{\kappa^2}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\mathcal{E}\left\{\Psi\mathbb{W} - \frac{1}{4}W^2\right\} + \text{ c.c. } + \frac{\xi}{\kappa^2}\int \mathrm{d}^{4|8}zEG^{ij}V_{ij},
\end{aligned}
\tag{218}
$$

where κ is the gravitational constant, ξ the cosmological constant, $\mathbb{W}$ is given by the expression (214), and V_{ij} is the Mezincescu prepotential. Within the projective-superspace approach of [56, 57, 59] , this action is equivalently given by (178) with the following Lagrangian [57]:

其中 κ 是引力常数，ξ 是宇宙学常数，$\mathbb{W}$ 由表达式 (214) 给出，V_{ij} 是梅赞斯库预势。在 [56, 57, 59] 的投影超空间方法中，该作用量等价地由 (178) 给出，带有如下拉格朗日量 [57]:

$$
\kappa^2\mathcal{L}^{(2)}_{\text{SUGRA}} = G^{(2)}\ln\frac{G^{(2)}}{\mathrm{i}\Upsilon^{(1)}\breve{\Upsilon}^{(1)}} - \frac{1}{2}V\Sigma^{(2)} + \xi VG^{(2)},
\tag{219}
$$

with V the tropical prepotential for the vector multiplet and Υ^{+} a weight-one arctic multiplet (both $\Upsilon^{(1)}$ and its smile-conjugate $\Upsilon^{(1)}$ are pure gauge degrees of freedom). The fact that the vector and the tensor multiplets are compensators means that their field strengths W and G^{ij} should possess non-vanishing expectation values, that is, $W \neq 0$ and $G \equiv \sqrt{\frac{1}{2}G^{ij}G_{ij}} \neq 0$.

其中 V 是向量多重态的热带预势，Υ^{+} 是权重为 1 的北极多重态 ($\Upsilon^{(1)}$ 及其微笑共轭 $\Upsilon^{(1)}$ 都是纯规范自由度)。向量和张量多重态作为补偿子，意味着它们的场强 W 和 G^{ij} 应当具有非零期望值，即 $W \neq 0$ 和 $G \equiv \sqrt{\frac{1}{2}G^{ij}G_{ij}} \neq 0$ 。

The equation of motion for the gravitational superfield [118, 127] is

引力超场 [118, 127] 的运动方程为

$$
G - W\bar{W} = 0,
\tag{220a}
$$

and it is consistent with the conditions $W \neq 0$ and $G \neq 0$. The equations of motion for the compensators are [118]

且该方程与条件 $W \neq 0$ 和 $G \neq 0$ 相容。补偿子的运动方程为 [118]

$$
\Sigma^{ij} - \xi G^{ij} = 0,
\tag{220b}
$$

$$
\mathbb{W} + \xi W = 0.
\tag{220c}
$$

The equations (220b) and (220c) can be degauged to U(2) superspace, which results in the following equations:

方程 (220b) 和 (220c) 可以退规范到 U(2) 超空间，得到如下方程:

$$\frac{1}{4}\left(\mathfrak{D}^{ij}+4S^{ij}\right)W=\frac{1}{4}\left(\overline{\mathfrak{D}}^{ij}+4\bar{S}^{ij}\right)\bar{W}=\xi G^{ij}, \tag{221a}$$

$$\frac{G}{8}\left(\overline{\mathfrak{D}}^{ij}+4\bar{S}^{ij}\right)\frac{G_{ij}}{G^{2}}=\xi W. \tag{221b}$$

The super-Weyl and local $\mathrm{U}(1)_R$ gauge freedom can be used to impose the gauge condition $W=\bar{W}=1$. The integrability conditions of these constraints are $\mathfrak{D}_A W=\mathfrak{D}_A\bar{W}=0$ which imply

超外尔和局域 $\mathrm{U}(1)_R$ 规范自由度可用来施加规范条件 $W=\bar{W}=1$ 。这些约束的可积性条件为 $\mathfrak{D}_A W=\mathfrak{D}_A\bar{W}=0$ ，由此推出

$$G_{\alpha\dot{\alpha}}{}^{ij}=0,\ \Phi_a=G_a. \tag{222}$$

After employing the redefinitions $\mathcal{D}^i_\alpha=\mathfrak{D}^i_\alpha$ and $\mathcal{D}_a=\mathfrak{D}_a-\mathrm{i}G_a\mathbb{Y}$, the resulting geometry coincides with SU (2) superspace for which the $\mathrm{U}(1)_R$ connection is pure gauge and can be set to zero. The equation of motion (220a) implies $G=1$ and $\mathcal{D}_A G=0$. The latter can be shown to imply the condition $\mathcal{D}_A G^{ij}=0$, which breaks the local $\mathrm{SU}(2)_R$ to a residual U (1) subgroup. It consists of those transformations that keep G^{ij} invariant. Integrability of the constraint $\mathcal{D}_A G^{ij}=0$ implies

在进行 $\mathcal{D}^i_\alpha=\mathfrak{D}^i_\alpha$ 和 $\mathcal{D}_a=\mathfrak{D}_a-\mathrm{i}G_a\mathbb{Y}$ 重定义后，得到的几何与 SU (2) 超空间一致，其中 $\mathrm{U}(1)_R$ 联络是纯规范，可以设为零。运动方程 (220a) 可推出 $G=1$ 和 $\mathcal{D}_A G=0$ 。可以证明后者会给出条件 $\mathcal{D}_A G^{ij}=0$ ，这将局域 $\mathrm{SU}(2)_R$ 破缺为剩余的 U (1) 子群。该子群由保持 G^{ij} 不变的变换构成。约束 $\mathcal{D}_A G^{ij}=0$ 的可积性推出

$$Y_{\alpha\beta}=0,\ G_{\alpha\dot{\alpha}}=0,\ S_{(i}{}^{k}G_{j)k}=0,\ \bar{S}_{(i}{}^{k}G_{j)k}=0. \tag{223}$$

The remaining supergravity equations (221) turn into

剩余的超引力方程 (221) 变为

$$S^{ij}=\bar{S}^{ij}=\xi G^{ij},\ S^2:=\frac{1}{2}S^{ij}S_{ij}=\xi^2,\ \mathcal{D}_A S^{ij}=0. \tag{224}$$

All the remaining information about the dynamics of supergravity is encoded in the super-Weyl tensor $W_{\alpha\beta}$.

超引力动力学的所有剩余信息都编码在超外尔张量 $W_{\alpha\beta}$ 中。

A maximally supersymmetric solution of (224) is characterized by the condition $W_{\alpha\beta}=0$ and the resulting superspace geometry is uniquely determined to be

(224) 的极大超对称解由条件 $W_{\alpha\beta}=0$ 刻画，得到的超空间几何唯一确定为

$$\left\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\right\} = 4S^{ij}M_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}S^{kl}J_{kl},\ \left\{\mathcal{D}_\alpha^i, \overline{\mathcal{D}}_j^k\right\} = -2\mathrm{i}\delta_j^i(\sigma^c)_\alpha{}^j\mathcal{D}_c, \tag{225a}$$

$$\left[\mathcal{D}_a, \mathcal{D}_\beta^j\right] = \frac{\mathrm{i}}{2}(\sigma_a)_{\beta\dot{\gamma}}S^{jk}\overline{\mathcal{D}}_k^{\dot{\gamma}},\ [\mathcal{D}_a, \mathcal{D}_b] = -S^2M_{ab}. \tag{225b}$$

This geometry corresponds to the four-dimensional $\mathcal{N} = 2$ AdS superspace

该几何对应四维 $\mathcal{N} = 2$ AdS 超空间

$$\mathrm{AdS}^{4|8} = \frac{\mathrm{OSp}\,(2\mid 4)}{\mathrm{SO}\,(3,1)\times \mathrm{SO}\,(2)}. \tag{226}$$

The most general $\mathcal{N} = 2$ supersymmetric nonlinear σ -models in AdS_4 were studied in [128-131]. They have important distinct features as compared with the $\mathcal{N} = 2$ supersymmetric nonlinear σ -models in Minkowski space. Specifically, the target space must be a non-compact hyperkähler manifold endowed with a Killing vector field which generates an SO(2) group of rotations on the two spheres of complex structures.

文献 [128-131] 研究了 AdS_4 中最一般的 $\mathcal{N} = 2$ 超对称非线性 σ 模型。与闵氏空间中的 $\mathcal{N} = 2$ 超对称非线性 σ 模型相比，这类模型具有重要的独特性质。具体而言，其目标空间必须是非紧致超凯勒流形，且配备有在复结构二维球上生成 SO(2) 旋转群的基灵矢量场。

As a generalization of (219), we consider the model for matter-coupled super-gravity [57]

作为 (219) 的推广，我们讨论物质耦合超引力模型 [57]

$$\kappa^2\mathcal{L}^{(2)} = -\frac{1}{2}V\sum^{(2)} + G^{(2)}\left(\ln\frac{G^{(2)}}{\mathrm{i}\breve{\Upsilon}^{(1)}\mathrm{e}^{-\xi V}\Upsilon^{(1)}} + \kappa^2K\left(\Upsilon^I, \breve{\Upsilon}^{\bar{J}}\right)\right), \tag{227}$$

where $K\left(\Phi, \overline{\Phi}\right)$ is the Kähler potential of a Kähler manifold and Υ^I are covariant weight-0 arctic multiplets.

其中 $K\left(\Phi, \overline{\Phi}\right)$ 是凯勒流形的凯勒势，Υ^I 是协变零权北极多重态。

Supergravity with Vector and Hyper-multiplet Compensators

带矢量补偿多重态与超补偿多重态的超引力

The compensators for this supergravity formulation are a vector multiplet and a polar hypermultiplet. The Lagrangian has the form

该超引力表述的补偿场为一个矢量多重态和一个极超多重态。拉格朗日量的形式为

$$\kappa^2\mathcal{L}^{(2)}_{\mathrm{SUGRA}} = -\frac{1}{2}V\sum^{(2)} - \mathrm{i}\breve{\Upsilon}^{(1)}\mathrm{e}^{-\xi V}\Upsilon^{(1)}, \tag{228}$$

with ξ a cosmological constant. The action is invariant under the gauge transformations

其中 ξ 为宇宙学常数。该作用量在规范变换下保持不变

$$\delta_\lambda V = \lambda + \breve{\lambda},\ \delta_\lambda \Upsilon^{(1)} = \xi \lambda \Upsilon^{(1)}. \tag{229}$$

The equation of motion for $\breve{\Upsilon}^{(1)}$ implies that

$\breve{\Upsilon}^{(1)}$ 的运动方程表明

$$\mathrm{e}^{-\xi V_+(\zeta)} \Upsilon^{(1)}(v) = \Upsilon^i v_i,\ V_+(\zeta) = \frac{1}{2} V_0 + \sum_{k=1}^{\infty} V_k \zeta^k, \tag{230}$$

where Υ^i obeys the equations

其中 Υ^i 满足方程

$$\underline{\nabla}_\alpha^{(i} \Upsilon^{j)} = 0,\ \overline{\underline{\nabla}}_{\dot\alpha}^{(i} \Upsilon^{j)} = 0, \tag{231}$$

which defines the on-shell Fayet-Sohnius hypermultiplet. Here $\underline{\nabla}_A$ denotes the gauge and conformal covariant derivative, which is obtained from ∇_A by adding a $\mathrm{U}(1)$ connection. We point out that $V(\zeta) = V_+(\zeta) + V_-(\zeta)$, where $V_-(\zeta)$ is the smile-conjugate of $V_+(\zeta)$. The equation of motion for V is

由此定义了壳 Fayet-Sohnius 超多重态。此处 $\underline{\nabla}_A$ 表示规范共变导数和共形共变导数，它由 ∇_A 添加一个 $\mathrm{U}(1)$ 联络得到。我们指出 $V(\zeta) = V_+(\zeta) + V_-(\zeta)$ ，其中 $V_-(\zeta)$ 是 $V_+(\zeta)$ 的微笑共轭。V 的运动方程为

$$\sum\nolimits^{(2)} - \xi \mathrm{i} \breve{\Upsilon}^{(1)} \mathrm{e}^{-\xi V} \Upsilon^{(1)} = 0 \Leftrightarrow \sum\nolimits^{ij} + \xi \mathrm{i} \overline{\Upsilon}^{(i} \Upsilon^{j)} = 0. \tag{232}$$

Finally, the equation of motion for the gravitational superfield H is (see [79,132] for the derivation)

最后，引力超场 H 的运动方程为 (推导见文献 [79,132])

$$\bar{W} W - \frac{1}{2} \overline{\Upsilon}_i \Upsilon^i = 0. \tag{233}$$

The supergravity-matter system (227) has a dual formulation [84] described by the Lagrangian

超引力-物质系统 (227) 存在由如下拉格朗日量描述的对偶表述 [84]

$$\kappa^2 \mathcal{L}^{(2)} = -\frac{1}{2} V \sum\nolimits^{(2)} - \mathrm{i} \breve{\Upsilon}^{(1)} \mathrm{e}^{-\xi V - \kappa^2 K(\Upsilon, \breve{\Upsilon})} \Upsilon^{(1)}. \tag{234}$$

If the cosmological constant vanishes, $\xi = 0$, this supergravity-matter system turns into the one introduced in [56].

若宇宙学常数为零，即 $\xi = 0$ ，该超引力-物质系统将退化为文献 [56] 中提出的系统

Conformal Supergravity, Topological Invariants, and Super-Weyl Anomalies

共形超引力、拓扑不变量与超外尔反常

In this section we describe a powerful formalism to generate locally superconformal higher-derivative invariants developed in [104]. Its applications include the super-field construction of the $\mathcal{N} = 2$ Gauss-Bonnet term and the general structure of super-Weyl anomalies in $\mathcal{N} = 2$ superconformal field theories. To start with, we review the $\mathcal{N} = 2$ conformal supergravity theory.

在本节中，我们介绍在文献 [104] 中发展的、用于生成局域超共形高导不变量的强大形式体系。该体系的应用包括 $\mathcal{N} = 2$ 高斯-博内项的超场构造，以及 $\mathcal{N} = 2$ 超共形场论中超外尔反常的一般结构。首先，我们回顾 $\mathcal{N} = 2$ 共形超引力理论。

Conformal Supergravity

共形超引力

The action for $\mathcal{N} = 2$ conformal supergravity [13] is

$\mathcal{N} = 2$ 共形超引力 [13] 的作用量为

$$S_{\mathrm{CSG}} = \frac{1}{4} \int \mathrm{d}^4 x \, \mathrm{d}^4\theta \mathcal{E} W^{\alpha\beta} W_{\alpha\beta} + \text{ c.c.} \tag{235}$$

The corresponding equation of motion is

对应的运动方程是

$$\nabla_{\alpha\beta} W^{\alpha\beta} = \overline{\nabla}^{\dot\alpha\dot\beta} \bar{W}_{\dot\alpha\dot\beta} = 0 \tag{236}$$

and states that the super-Bach multiplet (106a) vanishes. This equation is obtained by varying S_{CSG} with respect to a gravitational superfield $H = \bar{H}$ which is the only unconstrained prepotential of $\mathcal{N} = 2$ conformal supergravity modulo purely gauge degrees of freedom; see the discussions in [79, 104] and references therein. Performing this variation, we find

表明超巴赫多重态 (106a) 等于零。该方程通过对 S_{CSG} 关于引力超场 $H = \bar{H}$ 变分得到，$H = \bar{H}$ 是 $\mathcal{N} = 2$ 共形超引力中唯一无约束的预势，排除纯规范自由度；相关讨论参见文献 [79, 104] 及其中引文。完成该变分后我们得到

$$\delta_H \int \mathrm{d}^4 x \, \mathrm{d}^4\theta \mathcal{E} W^{\alpha\beta} W_{\alpha\beta} = 2 \int \mathrm{d}^{4|8} z E \delta H \nabla_{\alpha\beta} W^{\alpha\beta}, \tag{237}$$

where the variation δH is a real primary superfield of dimension -2 . Since the Bach multiplet $\mathfrak{B}$, Eq. (106a), and the variation δH are real, the functional

其中变分 δH 是维数为-2 的实基本超场。由于巴赫多重态 $\mathfrak{B}$ (式 106a) 和变分 δH 都是实的，该泛函

$$\mathfrak{P} = -\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mathrm{d}^4\theta \mathcal{E} W^{\alpha\beta} W_{\alpha\beta} + \text{ c.c.} \tag{238}$$

is a topological invariant being proportional to the Pontryagin term. As a consequence of (107a), the right-hand side of (237) is invariant under gauge transformations of the form [79, 104, 127]

是一个与庞特里亚金项成正比的拓扑不变量。作为 (107a) 的推论，(237) 的右侧在下述形式的规范变换下不变 [79, 104, 127]

$$\delta_\Omega H = \frac{1}{12}\nabla^{ij}\Omega_{ij} + \frac{1}{12}\bar{\nabla}_{ij}\bar{\Omega}^{ij}, \tag{239}$$

where the complex gauge parameter $\Omega_{ij} = \Omega_{ji}$ is unconstrained and has the superconformal properties

其中复规范参数 $\Omega_{ij} = \Omega_{ji}$ 无约束，且满足超共形性质

$$K^A\Omega_{ij} = 0,\ \mathbb{D}\Omega_{ij} = -3\Omega_{ij},\ \mathbb{Y}\Omega_{ij} = -2\Omega_{ij}. \tag{240}$$

This gauge invariance expresses the fact that the action (235) is locally superconfor-mal.

这种规范不变性表明作用量 (235) 是定域超共形的。

Any conformally flat superspace, $W_{\alpha\beta} = 0$, is a solution of Equation (236). It is instructive to linearize the conformal supergravity action (235) about such a background. From the linearized prepotential H , we construct the linearized super-Weyl tensor

任意共形平坦超空间 $W_{\alpha\beta} = 0$ 都是方程 (236) 的解。对共形超引力作用量 (235) 在这类背景下线性化是很有启发性的。我们从线性化预势 H 出发，构造线性化超外尔张量

$$\mathfrak{W}_{\alpha\beta} = \bar{\nabla}^4\nabla_{\alpha\beta}H \tag{241}$$

which is primary, $K^C\mathfrak{W}_{\alpha\beta} = 0$, and covariantly chiral, $\bar{\nabla}_i^{\dot{\alpha}}\mathfrak{W}_{\alpha\beta} = 0$. It proves to be invariant under the gauge transformations (239), $\delta_\Omega\mathfrak{W}_{\alpha\beta} = 0$, and obeys the Bianchi identity

它是基本的 $K^C\mathfrak{W}_{\alpha\beta} = 0$ ，且是协变手征的 $\bar{\nabla}_i^{\dot{\alpha}}\mathfrak{W}_{\alpha\beta} = 0$ 。可以证明它在规范变换 (239) $\delta_\Omega\mathfrak{W}_{\alpha\beta} = 0$ 下不变，且满足比安基恒等式

$$\nabla^{\alpha\beta}\mathfrak{W}_{\alpha\beta} = \bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{\mathfrak{W}}_{\dot{\alpha}\dot{\beta}}. \tag{242}$$

Thus, the action for linearized conformal supergravity is simply

因此，线性化共形超引力的作用量可简单写为

$$S_{\mathrm{LCSG}}=\frac{1}{4}\int \mathrm{d}^4x\ \mathrm{d}^4\theta\mathcal{E}\mathfrak{W}^{\alpha\beta}\mathfrak{W}_{\alpha\beta}+\ \text{c.c.} \tag{243}$$

If the background superspace is flat, the field strength (241) reduces to that described in [117], and the action (243) turns into the one given in [13, 117].

若背景超空间是平坦的，场强 (241) 就退化为文献 [117] 中描述的形式，作用量 (243) 则变为文献 [13, 117] 给出的形式。

The model (243) is known to possess U(1) duality invariance [133]. The formalism of $U(1)$ duality rotations has been used [133] to construct nonlinear extensions of (243).

已知模型 (243) 具有 U(1) 对偶不变性 [133]。文献 [133] 利用 $U(1)$ 对偶旋转的形式体系构造了 (243) 的非线性推广。

Logarithm Construction and the Gauss-Bonnet Invariant

对数构造与高斯-博内不变量

We now turn to describing the logarithm construction of [104] and its use in defining the $\mathcal{N}=2$ supersymmetric extension of the Gauss-Bonnet term.

我们现在来描述文献 [104] 中的对数构造方法，以及它如何用于定义高斯-博内项的 $\mathcal{N}=2$ 超对称扩展。

Let $\overline{\Phi}$ be a primary antichiral scalar with the superconformal properties:

设 $\overline{\Phi}$ 为具有如下超共形性质的主反手征标量:

$$K^A\overline{\Phi}=0,\ \nabla^i_\alpha\overline{\Phi}=0,\ \mathbb{D}\overline{\Phi}=w\overline{\Phi}\ \Rightarrow\ \mathbb{Y}\overline{\Phi}=2w\overline{\Phi}, \tag{244}$$

where $w\neq 0$, but it is otherwise arbitrary. We assume $\overline{\Phi}$ to be nowhere vanishing such that $\overline{\Phi}^{-1}$ exists. Then, it may be shown that $\overline{\nabla}^4\ln\overline{\Phi}$ is a primary chiral superfield of dimension 2,

其中 $w\neq 0$ 满足条件，除此之外任意。我们假设 $\overline{\Phi}$ 处处非零，因此 $\overline{\Phi}^{-1}$ 存在。可以证明，$\overline{\nabla}^4\ln\overline{\Phi}$ 是维数为 2 的主手征超场，

$$K^A\overline{\nabla}^4\ln\overline{\Phi}=0,\ \overline{\nabla}^{\dot\alpha}_i\overline{\nabla}^4\ln\overline{\Phi}=0,\ \mathbb{D}\overline{\nabla}^4\ln\overline{\Phi}=2\overline{\nabla}^4\ln\overline{\Phi}. \tag{245}$$

By following the degauging procedure to U(2) superspace, which was detailed in section ”U(2) Superspace,” it may be shown that

按照去规范过程处理到 U(2) 超空间 (该过程已在“U(2) 超空间”一节详细说明)，可以证明

$$\overline{\nabla}^4 \ln \overline{\Phi} = \bar{\Delta} \ln \overline{\Phi} + \frac{w}{2} \left(\bar{Y}^{\dot{\alpha}\dot{\beta}} \bar{Y}_{\dot{\alpha}\dot{\beta}} + \bar{S}_{ij} \bar{S}^{ij} + \frac{1}{6} \bar{\mathfrak{D}}_{ij} \bar{S}^{ij} \right) \equiv \bar{\Delta} \ln \overline{\Phi} + \frac{w}{2} \Xi,$$

(246)

where $\bar{\Delta}$ denotes the chiral projecting operator (145). It is important to note that since $\overline{\nabla}^4 \ln \overline{\Phi}$ and $\bar{\Delta} \ln \overline{\Phi}$ are both manifestly chiral, Ξ shares this property

其中 $\bar{\Delta}$ 表示手征投影算子 (145)。需要注意的是，由于 $\overline{\nabla}^4 \ln \overline{\Phi}$ 和 $\bar{\Delta} \ln \overline{\Phi}$ 都是明显手征的，因此 Ξ 也具备该性质

$$\bar{\mathfrak{D}}_i^{\dot{\alpha}} E = 0. \tag{247}$$

At the same time, we emphasize that while the right-hand side of (246) is primary, each individual term possesses an inhomogeneous contribution under the super-Weyl transformations of $U(2)$ superspace

同时我们要强调，尽管 (246) 的右侧是主超场，在 $U(2)$ 超空间的超外尔变换下，每个单独的项都存在非齐次贡献

$$\delta_{\sum} \bar{\Delta} \ln \overline{\Phi} = 2 \sum \bar{\Delta} \ln \overline{\Phi} + w \bar{\Delta} \sum, \; \delta_{\sum} \Xi = 2 \sum \Xi - 2 \bar{\Delta} \sum. \tag{248}$$

In the case of SU(2) superspace, these transformation laws turn into

在 SU(2) 超空间的情况下，这些变换律变为

$$\delta_\sigma \bar{\Delta} \ln \overline{\Phi} = 2\sigma \bar{\Delta} \ln \overline{\Phi} + w \bar{\Delta} \sigma, \; \delta_\sigma \Xi = 2\sigma \Xi - 2 \bar{\Delta} \sigma. \tag{249}$$

Our analysis leads to an important conclusion. Specifically, for every primary dimensionless chiral scalar Ψ , the following functional

我们的分析得出了一个重要结论: 对于任意无量纲主手征标量 Ψ ，如下泛函

$$\int \mathrm{d}^4 x \mathrm{d}^4 \theta \mathcal{E} \Psi \overline{\nabla}^4 \ln \overline{\Phi} = \int \mathrm{d}^{4|8} z E \Psi \ln \overline{\Phi} + \frac{w}{2} \int \mathrm{d}^4 x \mathrm{d}^4 \theta \mathcal{E} \Psi \Xi \tag{250}$$

is locally superconformal. Here the expression on the right is given in U(2) superspace (its form is preserved upon degauging to SU(2) superspace).

是局部超共形的。右侧的表达式在 U(2) 超空间中给出，去规范到 SU(2) 超空间后，其形式保持不变。

In Ref. [104] the superconformal chiral action

文献 [104] 中的超共形手征作用量

$$S_\chi^- = - \int \mathrm{d}^4 x \, \mathrm{d}^4 \theta \mathcal{E} \left(W^{\alpha\beta} W_{\alpha\beta} - 2 w^{-1} \overline{\nabla}^4 \ln \overline{\Phi} \right) \tag{251}$$

was identified with the $\mathcal{N}=2$ Gauss-Bonnet topological invariant. More precisely, it may be shown that, at the component level, S_χ^- is a combination of the Gauss-Bonnet and Pontryagin invariants. Under suitable boundary conditions on $\overline{\Phi}$, the functional (251) proves to be independent of $\overline{\Phi}$. This follows from (250) in conjunction with the identity in U (2) superspace

被识别为 $\mathcal{N}=2$ 高斯-博内拓扑不变量。更准确地说，可以证明，在分量层面，S_χ^- 是高斯-博内不变量与庞特里亚金不变量的组合。对 $\overline{\Phi}$ 施加合适的边界条件后，泛函 (251) 被证明与 $\overline{\Phi}$ 无关。该结论可由 (250) 结合 U (2) 超空间中的恒等式得到

$$\bar{\mathfrak{D}}_i^{\dot{\alpha}}\sigma=0 \Rightarrow \int \mathrm{d}^{4|8}z E\sigma=0, \tag{252}$$

for any covariantly chiral scalar σ. Therefore, we obtain

对任意协变手征标量 σ 成立，因此我们得到

$$S_\chi^-=-\int \mathrm{d}^4x\,\mathrm{d}^4\theta\mathcal{E}\left(W^{\alpha\beta}W_{\alpha\beta}-\Xi\right). \tag{253}$$

The topological nature of (251) was established in [104] at the component level. A solid superspace proof is still absent.

(251) 的拓扑性质已由文献 [104] 在分量层面证明，目前仍缺少完整的超空间证明。

Super-Weyl Anomalies

超外尔反常

Consider a superconformal field theory coupled to supergravity. The classical action of such a theory is invariant under the super-Weyl transformations, and it is independent of the supergravity compensators. In other words, the superconformal field theory couples to the Weyl multiplet.

考虑一个耦合超引力的超共形场论。这类理论的经典作用量在超外尔变换下不变，且与超引力补偿场无关。换句话说，超共形场论耦合于外尔多重态。

In the quantum theory, integrating out the matter fields leads to an effective action that is no longer a functional of the Weyl multiplet only. There are two different contributions to the $\mathcal{N}=2$ super-Weyl anomaly. One of them is given in terms of the supergravity multiplet. In the framework of SU(2) superspace, the super-Weyl variation of the effective action Γ has the form [134]

在量子理论中，积分掉物质场后得到的有效作用量不再只是外尔多重态的泛函。$\mathcal{N}=2$ 超外尔反常有两种不同贡献。其中一种由超引力多重态给出。在 SU(2) 超空间框架下，有效作用量 Γ 的超外尔变分形式如下 [134]

$$\delta_\sigma\Gamma=(c-a)\int \mathrm{d}^4x\,\mathrm{d}^4\theta\mathcal{E}\sigma W^{\alpha\beta}W_{\alpha\beta}+a\int \mathrm{d}^4x\,\mathrm{d}^4\theta\mathcal{E}\sigma\Xi+\text{ c.c.}, \tag{254}$$

for some anomaly coefficients a and c . One can check that the super-Weyl variation (254) obeys the Wess-Zumino consistency condition

由反常系数 a 和 c 给出。可以验证，超外尔变分 (254) 满足韦斯-祖米诺一致性条件

$$\left(\delta_{\sigma_1}\delta_{\sigma_2} - \delta_{\sigma_2}\delta_{\sigma_1}\right)\Gamma = 0. \tag{255}$$

This property guarantees the existence of Γ . The other sector of the $\mathcal{N} = 2$ super-Weyl anomaly is determined by local couplings in a superconformal field theory. According to [110, 135] , it is given by

该性质保证了 Γ 的存在。$\mathcal{N} = 2$ 超外尔反常的另一部分由超共形场论中的局域耦合决定。根据 [110, 135] ，其表达式为

$$\delta_\sigma \Gamma = \int \mathrm{d}^{4|8} z E \left(\sigma + \bar{\sigma}\right) K\left(X, \bar{X}\right), \tag{256}$$

where the Kähler potential $K\left(X, \bar{X}\right)$ is the same as in (166). Since the chiral scalars X^I are inert under the super-Weyl transformations, the anomaly clearly satisfies the Wess-Zumino consistency condition. The right-hand side of (256) is not invariant under Kähler transformations. However, the $\mathcal{N} = 2$ super-Weyl anomaly is invariant under a joint Kähler-Weyl transformation. A detailed analysis of the anomaly (256) is given in the original publications [110,135].

其中凯勒势 $K\left(X, \bar{X}\right)$ 与 (166) 中的相同。由于手征标量 X^I 在超外尔变换下不发生变化，该反常显然满足韦斯-祖米诺一致性条件。(256) 的右侧在凯勒变换下不不变，但 $\mathcal{N} = 2$ 超外尔反常在凯勒-外尔联合变换下不变。原始文献 [110,135] 中对反常 (256) 给出了详细分析。

The super-Weyl anomaly (254) is generated by the $\mathcal{N} = 2$ dilaton action [134]

超外尔反常 (254) 由 $\mathcal{N} = 2$ dilaton 作用量生成 [134]

$$S_{\mathrm{D}} = \frac{1}{4} f^2 \int \mathrm{d}^4 x\, \mathrm{d}^4 \theta \mathcal{E} \mathcal{L}^2 + \int \mathrm{d}^4 x\, \mathrm{d}^4 \theta \mathcal{E} \left\{(c - a)\, W^{\alpha\beta} W_{\alpha\beta} + a \Xi\right\} \ln \mathcal{Z} + \text{ c.c.}$$

$$+2a \int \mathrm{d}^{4|8} z E \ln \mathcal{Z} \ln \mathcal{Z}. \tag{257}$$

where f is a constant parameter, and $\mathcal{Z}$ is the chiral field strength of a vector multiplet such that $\mathcal{Z}^{-1}$ exists. One may check that the super-Weyl variation $\delta_\sigma S_{\mathrm{D}}$ coincides with the right-hand side of (254).

其中 f 是常数参数，$\mathcal{Z}$ 是向量多重态的手征场强，满足 $\mathcal{Z}^{-1}$ 存在的条件。可以验证，超外尔变分 $\delta_\sigma S_{\mathrm{D}}$ 与 (254) 的右侧完全一致。

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